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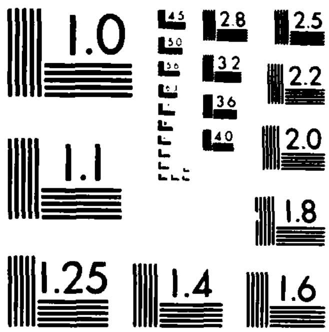

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Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

Point Processes Associated with Extreme Value Theory

by

Tailen Hsing

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Consider a stationary random sequence which satisfies certain dependence restrictions. The author studies the asymptotic behavior of a sequence of point processes that record the positions at which extreme values occur. Necessary and sufficient conditions are given for the weak convergence of the sequence. It is found that the usual Poisson limit when the random sequence is i.i.d. is replaced by a Compound Poisson limit. The asymptotic distributions of extreme order statistics are derived from the weak convergence result using simple combinatorial arguments.

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POINT PROCESSES ASSOCIATED WITH EXTREME VALUE THEORY

by

Tailen Hsing

A Dissertation submitted to the faculty of the University of North Carolina at Chapel Hill in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Statistics.

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CHAPTER I

INTRODUCTION

1.1 Extreme Value Theory and Point Processes

The focus of attention of classical extreme value theory is on the distributional properties of the maximum M_n of n independent and identically distributed random variables, as n becomes large. For example, the Extremal Types Theorem (cf. [21]) states: If for some constants $a_n > 0$, b_n, we have $P\{a_n(M_n - b_n) \le x\} \xrightarrow{w} G(x)$ for some non-degenerate G, then G is one of the following three extreme value types:

$$(1.1.1) \qquad \text{Type I:} \quad G(x) = \exp(-e^{-x}), \ -\infty < x < \infty;$$

$$(1.1.2) \qquad \text{Type II:} \quad G(x) = \begin{cases} 0 & x \le 0, \\ \exp(-x^{-\alpha}), \text{ for some } \alpha > 0, & x > 0; \end{cases}$$

$$(1.1.3) \qquad \text{Type III:} \quad G(x) = \begin{cases} \exp(-(-x)^{\alpha}), \text{ for some } \alpha > 0, & x \le 0, \\ 1 & x > 0. \end{cases}$$

It is natural to combine point processes with extreme value theory. Typically one is interested in the limit of a sequence of point processes obtained from extremal considerations, and it is often the case that a Poisson convergence result can be derived. For example, Resnick [31], Shorrock [35] and Pickands [30] all consider point processes involving "record times" in i.i.d. settings — a research direction which was initiated by Dwass' and Lamperti's work (cf. [10], [15]) on extremal processes.

Resnick [32] further noted that many results in this setting can be derived from a "Complete Poisson Convergence Theorem" in two dimensions.

It is known that the i.i.d. assumption can be relaxed. Leadbetter [18] considers a point process of exceedance positions under the conditions $D(u_n)$ and $D'(u_n)$ while Adler [1] generalizes Resnick's two dimensional result in [32] by assuming the conditions D and D'. In results of this kind, a long range dependence condition (e.g. $D(u_n)$) is used to give asymptotic independence of exceedances and together with local restriction (e.g. $D'(u_n)$) to avoid clustering of exceedances so that, in the limit, the point process under consideration performs just like one obtained from an i.i.d. sequence. If the local condition is weakened or omitted, then clustering of exceedances may occur. Some such situations have been considered. For example, Rootzén [33] studies the exceedance point process for a class of stable processes, Leadbetter [20] considers Poisson results for cluster centers, Mori [26] characterizes the limit of a sequence of point processes in two dimensions under strong mixing.

Our aim in this work is to study the limiting form of exceedence point processes (and of related but more complex point processes) under as broad assumptions as possible.

1.2 Framework and Poisson Results for I.I.D. Sequences

Let $\{\xi_j, j \in I\}$ be a strongly stationary sequence of random variables defined on some probability space $(\Omega, \mathcal{B}, P)$. Since we are mainly interested in the "weak" instead of the "strong" or "almost sure" type of results, the probability space will not be mentioned specifically each time. Write $M_n^{(k)}$ for the kth largest value of $\xi_1, \dots, \xi_n$, $k = 1,2,\dots$, and $M_n = M_n^{(1)}$. Let $F(u) = P\{\xi_1 \le u\}$ and $F_{i_1,i_2,\dots,i_k}(u) = P\{\xi_{i_j} \le u,\ j = 1,\dots,k\}$.

The following result is useful and suggestive despite being trivial.

Proposition 1.2.1 Let $\{\xi_j\}$ be an i.i.d. sequence. Let $0 \leq \tau \leq \infty$ and suppose that

(1.2.1) $$n[1 - F(u_n)] \to \tau \text{ as } n \to \infty.$$

Then

(1.2.2) $$P\{M_n \leq u_n\} \to e^{-\tau} \text{ as } n \to \infty.$$

Conversely, if (1.2.2) holds for some τ, $0 \leq \tau \leq \infty$, then so does (1.2.1).

Since this work will be centered upon point processes involving the sequence $\{u_n\}$ in (1.2.1), we write $\{u_n^{(\tau)}\}$, $\tau > 0$, for a sequence of constants which satisfies

(1.2.3) $$n[1 - F(u_n^{(\tau)})] \to \tau \text{ as } n \to \infty.$$

$\{u_n^{(\tau)}\}$ exists if and only if $\frac{1 - F(x-)}{1 - F(x)} \to 1$ as $x \to x_F \overset{\text{def}}{=} \sup\{x: F(x) < 1\}$ (cf. [21], Theorem 1.7.13). It is obvious that if $\{u_n^{(\tau)}\}$ exists for one $\tau > 0$, then it exists for all $\tau > 0$. We shall always assume implicitly that $\{u_n^{(\tau)}\}$ exists. For each $n = 1,2,\ldots$ and $\tau > 0$, define $N_n^{(\tau)}$ to be the point process (cf. Chapter II) on (0,1] that consists of points $\{j/n: 1 \leq j \leq n$ for which $\xi_j > u_n^{(\tau)}\}$. For convenience, $N_n^{(\tau)}$ will be referred to as the "exceedance point process". Now we state without proof a basic result which is again instructive.

Proposition 1.2.2 Let $\{\xi_j\}$ be an i.i.d. sequence and τ a constant in $(0,\infty)$. Then $N_n^{(\tau)}$ converges in distribution (cf. Chapter Two) to a Poisson Process on (0,1] with mean τ.

1.3 Poisson Results under $D(u_n)$ and $D'(u_n)$

$\{(\mu_1, \ldots, \mu_k) \in \Pi_1^k M(S_i): \Sigma_{i=1}^k \mu_i(I_{i1}) \leq t_1, \ldots, \Sigma_{i=1}^k \mu_i(I_{im}) \leq t_m\}$,

$m = 1, 2, 3, \ldots, I_{ij} \in T_i$, $i = 1, \ldots, k$, $j = 1, \ldots, m$. Since H is closed under finite intersections, we may conclude from a monotone class theorem (cf. [14], A2.1) that $D \supset \sigma(H)$. But by Lemma 2.2.5, $\sigma(H) \supset \Pi_1^k M(S_i)$. Therefore $P\{(\zeta_1, \ldots, \zeta_k) \in A\} = P\{(\eta_1, \ldots, \eta_k) \in A\}$ for each $A \in \Pi_1^k M(S_i)$; i.e., $(\zeta_1, \ldots, \zeta_k) \stackrel{d}{=} (\eta_1, \ldots, \eta_k)$, proving "(iii) $\Longrightarrow$ (i)". In similar manner, we can show "(ii)' $\Longrightarrow$ (i)" and this concludes the proof. Q. E. D.

2.3 Convergence in Distribution

Let S be a metric space and $P_0, P_1, P_2, \ldots$ be probability measures on $\mathcal{S}$, the Borel σ-field. P_n is said to converge weakly to P_0, or $P_n \Longrightarrow P_0$, if

$$\int_S f dP_n \to \int_S f dP_0$$

as $n \to \infty$ for every bounded continuous real function f on S. A family, π, of probability measures on $(S, \mathcal{S})$ is said to be relatively compact (or sequentially compact) if every subsequence contains a weakly convergent subsequence, π is said to be tight if for every $\varepsilon > 0$ there exists a compact set K such that $P(K) > 1 - \varepsilon$ for all P in π.

The following two results are among the most important.

Theorem 2.3.1 (The Portmanteau Theorem) Let P_n, P be probability measures on $(S, \mathcal{S})$. These five conditions are equivalent.

(i) $P_n \Rightarrow P$.

(ii) $\lim_{n\to\infty} \int f dP_n = \int f dP$ for all bounded, uniformly continuous real f.

(iii) $\limsup_n P_n(F) \leq P(F)$ for all closed F.

(iv) $\liminf_n P_n(G) \geq P(G)$ for all open G.

Theorem 2.2.6 Let $(\zeta_1, \ldots, \zeta_k)$ and $(\eta_1, \ldots, \eta_k)$ be two random elements in $(\Pi_1^k M(S_i), \Pi_1^k M(S_i))$ (or $(\Pi_1^k N(S_i), \Pi_1^k N(S_i))$) and let $T_i \subset B(S_i)$ be a semiring satisfying $\hat{\sigma}(T_i) = B(S_i)$, $i = 1, \ldots, k$. Then the following are equivalent.

(i) $(\zeta_1, \ldots, \zeta_k) \stackrel{d}{=} (\eta_1, \ldots, \eta_k)$;

(ii) $\Sigma_{i=1}^k \zeta_i f_i \stackrel{d}{=} \Sigma_{i=1}^k \eta_i f_i$, $(f_1, \ldots, f_k) \in \Pi_1^k F_c(S_i)$;

(ii)' $E\exp(-\Sigma_{i=1}^k \zeta_i f_i) = E\exp(-\Sigma_{i=1}^k \eta_i f_i)$, $(f_1, \ldots, f_k) \in \Pi_1^k F_c(S_i)$;

(iii) $(\Sigma_{i=1}^k \zeta_i(I_{i1}), \ldots, \Sigma_{i=1}^k \zeta_i(I_{im}))$

$\stackrel{d}{=} (\Sigma_{i=1}^k \eta_i(I_{i1}), \ldots, \Sigma_{i=1}^k \eta_i(I_{im}))$, $m = 1, 2, 3, \ldots$,

$I_{ij} \in T_i$, $i = 1, \ldots, k$, $j = 1, \ldots, m$.

Proof: We will prove this for random measures, the proof for point processes being similar. We proceed according to the following plan:

$$\begin{tikzcd}
(iii) \arrow[r, Rightarrow] & (i) \arrow[r, Rightarrow] & (ii) \arrow[d, Rightarrow] \\
 & & (ii)' \arrow[ul, Rightarrow]
\end{tikzcd}$$

It is obvious that (i) $\Rightarrow$ (ii) $\Rightarrow$ (ii)' and (i) $\Rightarrow$ (iii). Thus it suffices to show that both (ii)' and (iii) imply (i).

Suppose (iii) holds and define

$$D = \{A \in \Pi_{i=1}^k M(S_i): P\{(\zeta_1, \ldots, \zeta_k) \in A\} = P\{(\eta_1, \ldots, \eta_k) \in A\}\}.$$

Then D is closed under proper difference and monotone limits, and it contains $\Pi_1^k M(S_i)$. Further, since (iii) holds, D contains the class H of all sets of the form

product space $\Pi_1^k M(S_i) = \{(\mu_1, .., \mu_k): \mu_i \in M(S_i), i = 1, .., k\}$ with the usual product topology and σ-field. The following lemma is a simple consequence of [14], Lemma 1.4 and Lemma 4.1.

<u>Lemma 2.2.5</u> For each $i = 1, 2, .., k$, let $T_i \subset B(S_i)$ be a semiring with $\hat{\sigma}(T_i) = B(S_i)$. Then $\Pi_1^k M(S_i)$ (or $\Pi_1^k N(S_i)$) is generated by any one of the following three sets of mappings.

(i) $(\mu_1, .., \mu_k) \to \mu_i(B)$, $B \in \mathcal{B}(S_i)$, $i = 1, .., k$;

(ii) $(\mu_1, .., \mu_k) \to \mu_i(I)$, $I \in T_i$, $i = 1, .., k$;

(iii) $(\mu_1, .., \mu_k)$ $\mu_i f$, $f \in F_c(S_i)$, $i = 1, .., k$.

Let $\eta_1, \eta_2, .., \eta_k$ be random measures (resp., point processes) on S_1, .., S_k, respectively, defined on the same probability space (Ω, β, P). $(\eta_1, .., \eta_k)$ is therefore a random element in $(\Pi_1^k M(S_i), \Pi_1^k M(S_i))$ (resp., $(\Pi_1^k N(S_i), \Pi_1^k N(S_i)))$. $P(\eta_1, .., \eta_k)^{-1}$ is said to be the <u>distribution</u> of $(\eta_1, .., \eta_k)$. Two random elements $(\eta_1, .., \eta_k)$ and $(\zeta_1, .., \zeta_k)$ in $(\Pi_1^k M(S_i), \Pi_1^k M(S_i))$ (resp., $(\Pi_1^k N(S_i), \Pi_1^k N(S_i)))$ are said to be <u>equal in distribution</u>, or $(\eta_1, .., \eta_k) \overset{d}{=} (\zeta_1, .., \zeta_k)$ if $P(\eta_1, .., \eta_k)^{-1} \equiv P(\zeta_1, .., \zeta_k)^{-1}$. The function

$$L(f_1, .., f_k) = E\, e^{-\Sigma_{i=1}^k \eta_i f_i}$$

on $\Pi_1^k F(S_i) = \{(f_1, .., f_k): f_i \in F(S_i)\}$ is defined to be the <u>Laplace Transform</u> of $\underset{\sim}{\eta} = (\eta_1, .., \eta_k)$.

The following theorem provides a number of equivalent ways in which $P(\eta_1, .., \eta_k)^{-1}$ can be specified.

Proof: The assumption $P\{\Sigma_1^k \eta(E_j) < \infty\} > 0$ implies that $E\exp(-\Sigma_1^k t_i \eta(E_i))$ > 0, $t_1, t_2, .., t_k \in (0, \infty)$. Suppose first that $\eta(E_1), .., \eta(E_k)$ are independent. Then

$$0 < E\exp(-\Sigma_1^k t_i \eta(E_i)) = \Pi_1^k E\exp(-t_i \eta(E_i)),\ t_1, t_2, .., t_k > 0,$$

which implies by (2.2.1) that

$$\int_{N(S)\setminus\{o\}} [1 - \exp(-\Sigma_1^k t_i \mu(E_i))]\lambda(d\mu)$$

$$= \Sigma_1^k \int_{N(S)\setminus\{o\}} [1 - \exp(-t_i \mu(E_i))]\lambda(d\mu) < \infty,$$

or, equivalently,

(2.2.3) $\int_{M(S)\setminus\{o\}} \{\Sigma_1^k [1 - \exp(-t_i \mu(E_i))] - [1 - \exp(-\Sigma_1^k t_i \mu(E_i))]\}\lambda(d\mu) = 0.$

It is easy to see that for $\infty \geq x_1, x_2, .., x_k \geq 0$,

$$\Sigma_{i=1}^k (1 - e^{-x_i}) \geq 1 - e^{-\Sigma_1^k x_i}$$

with equality holds if and only if no more than one of the x_i is non-zero. The assertion (2.2.2) now follows from (2.2.3). The converse is similarly proved. Q.E.D.

A random measure (resp., point process) η is said to be <u>Compound Poisson</u> if it has a Laplace Transform $\exp[-\omega(1 - \pi\circ f)]$, where $\omega \in M(S)$ and π is the Laplace Transform of some probability measure π on $(0, \infty)$ (resp., $\mathbb{N}$). η is said to be <u>Poisson</u> with intensity ω if $\pi(\{1\}) = 1$. A Compound Poisson Process η on $\mathbb{R}^k$ (or a subset of $\mathbb{R}^k$), $k \in \mathbb{N}$, is said to be <u>homogeneous</u> if ω is a constant multiple of Lebesque measure. Throughout this work, we will be mainly concerned with homogeneous Compound Poisson Point Processes.

Now let $(S_i, \mathcal{S}_i)$, $i = 1, .., k$, be k Polish spaces, we can form the

$$(2.2.1) \qquad -\log E\exp(-\eta f) = \alpha f + \int_{M(S)\setminus\{o\}} [1 - \exp(-\mu f)]\lambda(d\mu)$$

defines a unique correspondence between the distributions of all infinitely divisible random measures η on S and the class of all pairs (α, λ), where $\alpha \in M(S)$ while λ is a measure on $M(S)\setminus\{o\}$ satisfying

$$\int_{M(S)\setminus\{o\}} (1 - e^{-\mu(B)})\lambda(d\mu) < \infty, \; B \in \mathcal{B}(S).$$

In the point process case, we have $\alpha = o$ while λ is confined to $N(S)\setminus\{o\}$.

We will call (2.2.1) the canonical representation of L_η, α and λ will be referred to as the canonical measures of η. The following results will be useful in Chapter 4.

Lemma 2.2.3 Let η be an infinitely divisible point process on $(S, \mathcal{S})$ with canonical measure λ, and E a set in $\mathcal{S}$. Then

$$P\{\eta(E) = 0\} = \exp(-\lambda\{\mu \in M(S)\setminus\{o\}: \mu(E) > 0\}).$$

Proof: It is readily seen from (2.2.1) that

$$\log E\, e^{-t\eta(E)} = -\int_{M(S)\setminus\{o\}} (1 - e^{-t\mu(E)})\lambda(d\mu), \; t > 0.$$

Passing t to ∞, the conclusion follows by monotone convergence. Q.E.D.

Lemma 2.2.4 Let η be an infinitely divisible point process on $(S, \mathcal{S})$ with canonical measure λ. Suppose $E_1, E_2, \ldots, E_k$ are sets in $\mathcal{S}$ such that $P\{\Sigma_1^k \eta(E_i) < \infty\} > 0$. Then $(E_1), \ldots, \eta(E_k)$ are mutually independent if and only if for i, j satisfying $1 \le i < j \le k$,

$$(2.2.2) \qquad \lambda\{\mu \in M(S)\setminus\{o\}: \mu(E_i) > 0, \mu(E_j) > 0\} = 0$$

$$\mu \to \mu f, \ f \in F_c(S)$$

are continuous is said to be the vague topology. Let M(S) be equipped with the vague topology and the Borel σ-field. N(S) is known to be vaguely closed in M(S) (cf. [14], A7.4). Let N(S) be equipped with the relative topology and σ-field. Then, it is known that M(S) and N(S) are both Polish (cf. [14], A7.7).

A <u>random measure</u> (resp., <u>point process</u>) η is a measurable mapping from some probability space $(\Omega, \mathcal{B}, P)$ into $(M(S), \mathcal{M}(S))$ (resp., $(N(S), \mathcal{N}(S))$). $P\eta^{-1}$, the probability measure on $(M(S), \mathcal{M}(S))$ induced by η, is called the distribution of η. Write $\mathcal{B}_\eta = \{B \in \mathcal{B}(S) : \eta(\partial B) = 0 \text{ a.s.}\}$. For $f \in F(S)$, let ηf be the random variable defined by $\eta f(\omega) = \int_S f d\eta(\omega)$, $\omega \in \Omega$. Just as in the case of random variables, we can define Laplace Transforms for random measures (or point processes). The <u>Laplace Transform</u> for η, denote by $L_\eta(f)$, is a function on (S) defined by

$$L_\eta(f) = \exp(-\eta f) = \exp(-\int_S f d\eta).$$

As we shall see in Lemma 2.2.2, $P\eta^{-1}$ is completely determined by $L_\eta(f)$.

A random measure (resp., point process) η is said to be <u>infinitely divisible</u> if for each $n \in \mathbb{N}$, there exists some independent and identically distributed random measures (resp., point processes) $\eta_1, \eta_2, \ldots, \eta_n$ such that

$$\eta \overset{d}{=} \eta_1 + \eta_2 + \ldots + \eta_n.$$

The following result is important.

<u>Theorem 2.2.2</u> (cf. [14], Theorem 6.1) The relation

of all functions in $F(S)$ which are continuous and have compact supports. Let $\mathcal{B}(S)$ be the ring that consists of all the bounded (relatively compact) sets in S. A semiring $\mathcal{T} \subset \mathcal{B}(S)$ is said to be a DC-semiring (D for dissecting, C for covering) if $\mathcal{T}$ is a semiring with the property that given any $B \in \mathcal{B}(S)$ and any $\varepsilon > 0$, there exists some finite cover of B composed of $\mathcal{T}$-sets of diameters less than ε (in any fixed metrization). The notion of DC-semiring is independent of the choice of metric (cf. Lemma 1.1 of [14]). For any collection $\mathcal{U}$ of sets in $\mathcal{B}(S)$, $\hat{\sigma}(\mathcal{U})$ denotes the smallest ring which contains all the sets in $\mathcal{U}$ and all the bounded sets of the form $\cap_1^\infty B_i$, $B_i \in \mathcal{U}$. If $\mathcal{T} \subset \mathcal{B}(S)$ is a DC-semiring, then $\hat{\sigma}(\mathcal{T}) = \mathcal{B}(S)$ (cf. [14], Lemma 1.2).

A measure μ on $(S, \mathcal{S})$ is said to be locally finite if $\mu(B) < \infty$ for all $B \in \mathcal{B}(S)$. Write $\mu f = \int_S f d\mu$, $f \in F(S)$. Let δ_s, $s \in S$, denote the measure with a unit mass at s, and o the null measure on S. The structure of S provides the following decomposition for locally finite measures.

Lemma 2.2.1 (cf. [14], Lemma 2.1) Any locally finite measure μ on $(S, \mathcal{S})$ can be written in the form

$$\mu = \mu_d + \Sigma_{j=1}^{k} b_j \delta_{t_j}$$

for some diffuse (or non-atomic) measure μ_d, $k \in I_+ \cup \{\infty\}$, $b_1, b_2, \ldots \in (o, \infty)$ and $t_1, t_2, \ldots \in S$. This decomposition is unique apart from the order of terms, provided that the t_j are assumed to be distinct. μ is integer valued if and only if $\mu_d = 0$ and $b_1, b_2, \ldots \in \mathbb{N}$.

A sequence of measures $\{\mu_n\}_1^\infty$ in $M(S)$ is said to converge vaguely to a measure $\mu \in M(S)$ if $\lim_n \mu_n f = \mu f$ for each $f \in F_c(S)$. The coarsest topology on $M(S)$ with respect to which all the mappings

CHAPTER II

RANDOM MEASURES AND POINT PROCESSES

2.1 Introduction

Point processes were first studied in the contexts of telephone traffic models and queueing models, where, typically, a point process refers to a random step function on the line representing the number of "customers" in the "system" as time progresses. Along with the other advances in probability (e.g. the theory of weak convergence), the theory of point processes on the line was extended to the general settings of random measures on abstract spaces. A brief history of the development of the theory can be found in [14].

For introductory purpose, [14] and [23] both provide rather complete accounts of the theory with rigour and elegance, but with different emphases and approaches. However, some of the results there are too general to be applied directly for our purpose. Thus the aim of this chapter is to introduce the very basic notions of random measures and point processes, and to present results that are specially tailored (mainly from those in [14]) for later used.

2.2 Basic Framework

Let S be a topological space with a separable and complete metrization, such a space is said to be <u>Polish</u>. In S we introduce the Borel σ-field $\mathcal{S}$, i.e., the σ-field generated by the topology. $F(S)$ will denote the class of all $\mathcal{S}$-measurable functions that are non-negative, and $F_c(S)$ the sub-class

Conversely if $\lim_{n\to\infty} k_n P\{M_{r_n} > u_n^{(\tau)}\} = \theta\tau$, then

$$P\{M_{r_n} \le u_n^{(\tau)}\} = 1 - \theta\tau/k_n[1 + o(1)]$$

so that

$$P^{k_n}\{M_{r_n} \le u_n^{(\tau)}\} = [1 - \theta\tau/k_n + o(1/k_n)]^{k_n} \to e^{-\theta\tau}$$

and hence $P\{M_n \le u_n^{(\tau)}\} \to e^{-\theta\tau}$ by Lemma 1.3.1. Q. E. D.

By arguments similar to those used in proving Theorem 1.3.3, one can have a result concerning the convergence of N_n^* (cf. [20]).

<u>Theorem 1.5.4</u> Let the stationary sequence $\{\xi_j\}$ satisfy $D(u_n^{(\tau)})$ for some $\tau > 0$ and let the sequence $\{k_n\}$ satisfy (1.5.1) ~ (1.5.3). Then N_n^* converges in distribution to a Poisson Process on (0, 1] with intensity parameter $\theta\tau$.

Finally note that, under the assumptions of the preceding theorem, the mean "cluster size" of exceedances of $u_n^{(\tau)}$ is given by

$$(1.5.4) \quad E(\Sigma_{j=1}^{r_n} \chi_{n,j}^{(\tau)} | \Sigma_{j=1}^{r_n} \chi_{n,j}^{(\tau)} > 0) = E(\Sigma_{j=1}^{r_n} \chi_{n,j}^{(\tau)})/P\{\Sigma_{j=1}^{r_n} \chi_{n,j}^{(\tau)} > 0\}$$

$$= \Sigma_{j=1}^{r_n} E(\chi_{n,j}^{(\tau)})/P\{M_{r_n} > u_n^{(\tau)}\} \sim (r_n\tau/n)/(\theta\tau/k_n),$$

which converges to θ^{-1} as $n \to \infty$. This implies that θ^{-1} is the asymptotic mean cluster size, providing an intriguing interpretation for θ.

It is intuitively plausible that one may be able to prove a Compound Poisson result for the exceedances themselves rather than cluster positions under suitable assumptions. This is one of the major goals of this work.

positions of the clusters. For this purpose alone, the choice of $\{k_n\}$ requires that r_n be large compared with all the cluster sizes so that a cluster of exceedances does not get counted more than once, and, on the other hand, r_n should be small so that the positions of the clusters can be recorded accurately. Together with the consideration concerning the mixing condition $D(u_n)$, it will be seen that an appropriate $\{k_n\}$ is one which satisfies

(1.5.1) $$k_n \to \infty,$$

(1.5.2) $$k_n \ell_n / n \to 0,$$

(1.5.3) $$k_n \alpha_{n,\ell_n} \to 0.$$

where $\alpha_{n,\ell}$ and ℓ_n are the usual constants used in stating $D(u_n)$. The existence of such a sequence is trivial.

<u>Lemma 1.5.3</u> Let the stationary sequence $\{\xi_j\}$ satisfy $D(u_n^{(\tau)})$ for some $\tau > 0$ and let $\{k_n\}$ be a sequence which satisfies (1.5.1) ~ (1.5.3). Then $\lim_{n\to\infty} P\{M_n \le u_n^{(\tau)}\} = e^{-\theta\tau}$ for some $\theta \in [0, 1]$ if and only if $\lim_{n\to\infty} k_n P\{M_{r_n} > u_n^{(\tau)}\} = \theta\tau$ where $r_n = [n/k_n]$.

<u>Proof:</u> Suppose $\lim_{n\to\infty} P\{M_n \le u_n^{(\tau)}\} = e^{-\theta\tau}$. Then Lemma 1.3.1 implies that $\lim_{n\to\infty} P^{k_n}\{M_{r_n} \le u_n^{(\tau)}\} = e^{-\theta\tau}$. It follows simply that $P\{M_{r_n} > u_n^{(\tau)}\} \to 1$ and

$$\log(1 - P\{M_{r_n} > u_n^{(\tau)}\}) = -\theta\tau/k_n[1 + o(1)]$$

so that

$$-P\{M_{r_n} > u_n^{(\tau)}\} [1 + o(1)] = -\theta\tau/k_n[1 + o(1)]$$

giving $\lim_{n\to\infty} k_n P\{M_{r_n} > u_n^{(\tau)}\} = \theta\tau$ as reguired.

totic distribution of M_n (or, more generally, the kth largest maxima). For example, the extremal type theorems characterize the possible types of limit laws that M_n can have under linear normalization. Although a vast number of distributions belong to the domain of attraction (cf. [21], Theorem 1.6.2) of the three extreme value type distributions, our study of extreme value theory should by no means be confined to linear normalizations. The possibly non-linear function $u_n^{(\cdot)}$ provides perhaps the most "accessible" non-linear normalization. Suppose that $\{\xi_j\}$ has extremal index θ, and that for each n, $u_n^{(\cdot)}$ is strictly decreasing. Then

$$P\{u_n^{-1}(M_n) \leq x\} \to 1 - e^{-\theta x}, \ x > 0.$$

However, it does not generalize the linear normalization as there are cases where linear normalizations are applicable while $u_n^{(\cdot)}$ may not even be defined. While this study is based on the normalization $u_n^{(\cdot)}$, most of the results are expected to be extended to more general settings.

1.5 Point Process of Cluster Positions

It is of interest to explore the limiting behavior of $N_n^{(\tau)}$ when the extremal index is not necessarily 1. In this case, the limiting distribution of $N_n^{(\tau)}$, when it exists, may be a cluster process instead of a Poisson Process, as was illustrated by Example 1.4.1.

Leadbetter [20] studies the process of cluster positions under $D(u_n)$ as follows. First devide the integers $1,2,\ldots,n$ into k_n intervals, with $\{k_n\}$ properly chosen. Let N_n^* be the point process which consists of points $\{j/k_n\colon j = 1,\ldots,k_n$ for which $\Sigma_{i=(j-1)r_n+1}^{jr_n} 1(\xi_j > u_n^{(\tau)}) > 0\}$ where $r_n = [n/k_n]$. That is, any group of exceedances in the interval $[(j-1)r_n+1, jr_n]$ is regarded as a cluster and replaced by a single point at j/k_n. One can therefore think of N_n^* as a devise that records the approximate

Simple calculations show that $\lim_{n\to\infty} P\{M_n \leq u_n^{(\tau)}\} = e^{-\tau/2}$. It is simply seen that $N_n^{(\tau)}$ does not converge in distribution to a Poisson Process since exceedances always occur in pairs.

Loynes [22] proves that, under strong-mixing, the only possible limit functions of $P\{M_n \leq u_n^{(\tau)}\}$ are $e^{-\theta\tau}$, where $\theta \in [0, 1]$. The following theorem due to Leadbetter (cf. [20]) generalizes Loynes' result (and a result of O'Brien [27]).

<u>Theorem 1.4.2</u> Let $\{\xi_j\}$ be a stationary sequence and $\{u_n^{(\tau)}\}$ constants satisfying (1.2.3) and such that $D(u_n^{(\tau_0)})$ holds for some $\tau_0 > 0$. Then there exist constancs θ, θ', $0 \leq \theta \leq \theta' \leq 1$ such that

$$\limsup_{n\to\infty} P\{M_n \leq u_n^{(\tau)}\} = e^{-\theta\tau}$$

$$\liminf_{n\to\infty} P\{M_n \leq u_n^{(\tau)}\} = e^{-\theta'\tau}$$

for $0 < \tau \leq \tau_0$. Hence if $P\{M_n \leq u_n^{(\tau)}\}$ converges for some τ, $0 < \tau \leq \tau_0$, then $\theta = \theta'$ and $P\{M_n \leq u_n^{(\tau)}\} \to e^{-\theta\tau}$ for all such τ.

We shall say (cf. [20]) that $\{\xi_j\}$ has <u>extremal index</u> θ, $\theta \in [0, 1]$, if for each $\tau > 0$, $\{u_n^{(\tau)}\}$ exists and $P\{M_n \leq u_n^{(\tau)}\} \to e^{-\theta\tau}$ as $n \to \infty$.

With this definition, the case where $D(u_n^{(\tau)})$ and $D'(u_n^{(\tau)})$ both hold leads to the extremal index $\theta = 1$. The sequence in Example 1.4.1 has extremal index $\theta = 1/2$. Many authors (see Leadbetter [20] and the reference therein) have exhibited illuminating examples concerning the extremal index. Here we only mention that for each $\theta \in [0, 1]$, there exist sequences that have θ as their extremal indices and that there are examples for which the extremal indices do not exist.

It is worth comparing the normalization $u_n^{(\cdot)}$ with the more traditional linear normalization. Practically, we are often interested in the asymp-

The condition $D'(u_n)$ will be said to hold if $\limsup_{n \to \infty} n \sum_{j=2}^{[n/k]} P\{\xi_1 > u_n, \xi_j > u_n\} \to 0$ as $k \to \infty$. The following result is trivial (but useful) for i.i.d. random variables and is also basic in a study of dependent cases.

Theorem 1.3.2 Let $\{u_n\}$ be constants such that $D(u_n)$ and $D'(u_n)$ hold for stationary sequence $\{\xi_j\}$. Let $0 \le \tau < \infty$. Then $P\{M_n \le u_n\} \to e^{-\tau}$ if and only if $n[1 - F(u_n)] \to \tau$.

It may be shown (cf. [18]), for example, by using a general point process theorem of Kallenberg, that the following result holds.

Theorem 1.3.3 Let $\tau \in (0, \infty)$ be fixed and suppose that $D(u_n^{(\tau)})$ and $D'(u_n^{(\tau)})$ hold for the stationary sequence $\{\xi_j\}$. Then $N_n^{(\tau)}$ converges in distribution to a Poisson Process N on (0, 1] with parameter τ.

Intuitively, the condition $D(u_n)$ provides the independence associated with the occurrence of events in a Poisson Process while $D'(u_n)$ limits the possibility of clustering of exceedances so that multiple events are excluded in the limit.

It should be noted that Theorem 1.3.2 is an improvement of both Loynes' and Berman's results.

1.4 Relaxation of $D'(u_n)$ and the Extremal Index

The theory under $D(u_n)$ and $D'(u_n)$ is elegant indeed; however, a great many processes do not satisfy $D'(u_n)$ as the following example shows.

Example 1.4.1 Suppose $\{X_j\}_1^\infty$ is an i.i.d. sequence with marginal distribution U(0, 1). Let $\{\xi_j\}_1^\infty$ be defined by

$$\xi_j = \max(X_j, X_{j+1}), \; j = 1,2,\ldots$$

It is well known that the independence assumptions in Proposition 1.2.1 and 1.2.2 are far from being necessary. Loynes [22] gives a sufficient condition for the equivalence of (1.2.1) and (1.2.2) when $\{\xi_j\}$ is strongly mixing, i.e., when $\alpha(\ell) \stackrel{\text{def}}{=} \sup(|P(A \cap B) - P(A)P(B)|: A \in F_{-\infty}^{0}, B \in F_{\ell}^{\infty}) \to 0$ as $\ell \to \infty$ where $F_{-\infty}^{0} = \sigma(\xi_j, j \leq 0)$, $F_{\ell}^{\infty} = \sigma(\xi_j, j \geq \ell)$. Berman [3] considers the specific case where $\{\xi_j\}$ is a Gaussian sequence and shows that "$r_n \log n \to 0$" is sufficient for "(1.2.1) $\Longleftrightarrow$ (1.2.2)", where r_n is the covariance function.

Leadbetter [17] introduces a "Distributional Mixing" approach, which we now briefly describe. The condition $D(u_n)$ will be said to hold if for any integers $1 \leq i_1 < \dots < i_p < j_1 < \dots < j_{p'} \leq n$ for which $j_1 - i_p \geq \ell$, we have

$$|F_{i_1 \dots i_p, j_1, \dots, j_{p'}}(u_n) - F_{i_1 \dots i_p}(u_n) F_{j_1 \dots j_{p'}}(u_n)| \leq \alpha_{n,\ell}$$

where $\alpha_{n,\ell_n} \to 0$ as $n \to \infty$ for some sabsequence $\ell_n = o(n)$. This is a long range dependence restriction of the same type as strong mixing but significantly weaker. Using a technique first used by Loynes, one can prove the following result which shows that, roughly, the maxima on propely chosen subintervals are asymptotically independent under the condition $D(u_n)$.

<u>Lemma 1.3.1</u> Let $\{u_n\}$ be a sequence of constants and let $D(u_n)$ be satisfied by the stationary sequence $\{\xi_n\}$. Let $\{k_n\}$ be a sequence of constants such that $k_n = o(n)$ and, in the notation used in stating $D(u_n)$, $k_n \ell_n = o(n)$, $k_n \alpha_{n,\ell_n} \to 0$. Then

$$P\{M_n \leq u_n\} - P^{k_n}\{M_{r_n} \leq u_n\} \to 0 \text{ as } n \to \infty$$

where $r_n = [n/k_n]$.

(v) $\lim_{n\to\infty} P_n(A) = P(A)$ for all A such that $P(\partial A) = 0$.

Theorem 2.3.2 (Prohorov) A family π of probability measures on $(S, \mathcal{S})$ is relatively compact provided it is tight. The converse is also ture if S is separable and topologically complete.

Suppose X_0, X_1, X_2, ... are random elements (not necessarily defined on the same probability space) in S. X_n is said to converge in distribution to X_0, or $X_n \xrightarrow{d} X_0$, if P_n, the probability measure induced by X_n, converges weakly to P_0, the probability measure induced by X_0. The notions of tightness and relative compactness for random elements are similarly defined in terms of the induced measures. See [5] for the proofs of Theorem 2.3.1 and 2.3.2, and a fuller account of the theory of weak convergence.

We now specialize to random measures and point processes. First note that since N(S) is closed in M(S), it is easily seen from Theorem 2.3.1 (iii) that the limit of a sequence of point processes is itself a point process. Since point processes may be regarded as random elements in either M(S) or N(S), we have two notions of convergence for them. However, using "restriction" and "extention" mappings, it follows from [5], Theorem 5.1 that the two are in fact equivalent.

Lemma 2.3.3 A sequence of random elements $\{(\eta_{n1}, \ldots, \eta_{nk})\}_{n=1}^{\infty}$ in $(\Pi_1^k M(S_i), \Pi_1^k M(S_i))$ (or $(\Pi_1^k N(S_i), \Pi_1^k N(S_i))$) is relatively compact if and only if

(2.3.1) $\lim_{t\to\infty} \limsup_{n\to\infty} P\{\eta_{ni}(B_i) > t\} = 0$, $B_i \in \mathcal{B}(S_i)$, for each $i = 1,2,\ldots,k$,

or if and only if

(2.3.2) $\lim_{t\to\infty} \limsup_{n\to\infty} P\{\Sigma_{i=1}^k \eta_{ni}(B_i) > t\} = 0$, $B_i \in \mathcal{B}(S_i)$, $i = 1,2,\ldots,k$.

Proof: Lemma 4.5 of [14] together with the fact (cf. [13]) that $\{(\eta_{n1}, \ldots, \eta_{nk})\}_{n=1}^{\infty}$ is relatively compact iff $\{\eta_{n1}\}$, $\{\eta_{n2}\}$, .., $\{\eta_{nk}\}$ are imply the first assertion. Suppose (2.3.1) holds. Then

$$\lim_{t\to\infty} \limsup_{n\to\infty} P\{\Sigma_{i=1}^{k}\eta_{ni}(B_i) > t\}$$

$$\le \Sigma_{i=1}^{k} \lim_{t\to\infty} \limsup_{n\to\infty} P\{\eta_{ni}(B_i) > t/r\}$$

$$= 0,$$

which shows that (2.3.2) holds. Suppose conversely that (2.3.2) holds. Then (2.3.1) holds trivially since $P\{\eta_{ni}(B_i) > t\} = P\{\Sigma_{j=1}^{k}\eta_{nj}(B_j) > t\}$ with $B_j = B_i$ or ϕ, the empty set, depending on $j = i$ or not. Q. E. D.

The next result is an analogue of the so-called continuous mapping theorem (cf. [5], Theorem 5.1).

Lemma 2.3.4 Let $(\eta_1, \ldots, \eta_k)$, $(\eta_{11}, \ldots, \eta_{1k})$, $(\eta_{21}, \ldots, \eta_{2k})$, ... be random elements in $(\Pi_1^k M(S_i), \Pi_1^k M(S_i))$ (or $(\Pi_1^k N(S_i), \Pi_1^k N(S_i))$ and suppose that

$$(\eta_{n1}, \ldots, \eta_{nk}) \xrightarrow{d} (\eta_1, \ldots, \eta_k).$$

Let m be any positive integer. For each i = 1, 2, .., k, let f_{ij}, j = 1, 2, .., m, be bounded measurable functions on S_i with bounded supports and satisfy $\eta_i(D_{f_{ij}}) = 0$ a.s., where D_f is the set of discontinuity points of f. Then

$$(\Sigma_{i=1}^{k}\eta_{ni}f_{i1}, \ldots, \Sigma_{i=1}^{k}\eta_{ni}f_{im}) \xrightarrow{d} (\Sigma_{i=1}^{k}\eta_i f_{i1}, \ldots, \Sigma_{i=1}^{k}\eta_i f_{im}).$$

Proof: Suppose first that all the f_{ij} are non-negative. Let π be the mapping

$$(\mu_1, \ldots, \mu_k) \to (\Sigma_{i=1}^{k}\mu_i f_{i1}, \ldots, \Sigma_{i=1}^{k}\mu_i f_{im})$$

and π_j the mapping

$$(\mu_1, \ldots, \mu_k) \to \Sigma_{i=1}^{k}\mu_i f_{ij}, \ j = 1, 2, \ldots, m.$$

Obviously $D_\pi \subset \cup_{j=1}^{m} D_{\pi_j}$. Further, applying [14], A7.3,

$$D_{\pi_j} \subset \cup_{i=1}^{k}\{(\mu_1, \ldots, \mu_k): \mu_i(D_{f_{ij}}) > 0\}.$$

Therefore, by Boole's inequality,

$$P(D_\pi) \le \Sigma_{j=1}^{m}\Sigma_{i=1}^{k} P\{\eta_i(D_{f_{ij}}) > 0\},$$

which equals zero by assumption. The assertion follows from [5], Theorem 5.1. Suppose now the f_{ij} are not necessarily non-negative. Then treat the positive and negative parts seperately, and the result follows again by Theorem 5.1 of [5]. Q. E. D.

Next, we look at a result that contains some of the basic (and powerful) tools for proving convergence of random measures (and point processes).

<u>Theorem 2.3.5</u> Let $(\eta_1, \ldots, \eta_k)$, $(\eta_{n1}, \ldots, \eta_{nk})$, $n = 1, 2, 3, \ldots$ be random elements in $\Pi_1^k M(S_i)$ (or $\Pi_1^k N(S_i)$) and let $\mathcal{T}_i \subset \mathcal{B}_{\eta_i}$, $i = 1, \ldots, k$, be DC-semirings (see section 2 for definition) in $S_1, \ldots, S_k$ respectively. Then the following are equivalent.

(i) $(\eta_{n1}, \ldots, \eta_{nk}) \xrightarrow{d} (\eta_1, \ldots, \eta_k)$,

(ii) $\Sigma_{i=1}^{k}\eta_{ni} f_i \xrightarrow{d} \Sigma_{i=1}^{k}\eta_i f_i$, $(f_1, \ldots, f_k) \in \Pi_1^k \mathcal{F}_c(S_i)$,

(ii)' $E\exp(-\Sigma_{i=1}^{k}\eta_{ni} f_i) \to E\exp(-\Sigma_{i=1}^{k}\eta_i f_i)$, $(f_1, \ldots, f_k) \in \Pi_1^k \mathcal{F}_c(S_i)$,

(iii) $\quad (\Sigma_{i=1}^{k}\eta_{ni}(B_{i1}), \ldots, \Sigma_{i=1}^{k}\eta_{ni}(B_{im})) \xrightarrow{d} (\Sigma_1^k\eta_i(B_{i1}), \ldots, \Sigma_1^k\eta_i(B_{im})),$

$m = 1, 2, 3, \ldots, B_{ij} \in \mathcal{T}_i, i = 1, \ldots, k, j = 1, \ldots, m.$

Proof: By Lemma 2.3.4 (i) implies (ii) and (iii) while (ii) implies (ii)'. Thus it suffices to show that (ii)' $\Longrightarrow$ (ii) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (i). Suppose (ii)' holds. Then

$$E\exp(-t\Sigma_{i=1}^{k}\eta_{ni}f_i) \to E\exp(-t\Sigma_{i=1}^{k}\eta_i f_i),\ t > 0,$$

showing (ii). Suppose now (ii) holds. For $B_i \in \mathcal{B}(S_i)$, $i = 1, \ldots, k$, we can find some $f_i \in \mathcal{F}_c(S_i)$ with $f_i \geq 1_{B_i}$ (cf. [14], A6.1) where 1_{B_i} is the indicator of the set B_i. Then

$$\lim_{t\to\infty} \limsup_{n\to\infty} P\{\Sigma_{i=1}^{k}\eta_{ni}(B_i) > t\}$$

$$\leq \lim_{t\to\infty} \limsup_{n\to\infty} P\{\Sigma_{i=1}^{k}\eta_{ni}f_i > t\}$$

$$\leq \lim_{t\to\infty} \limsup_{n\to\infty} P\{\Sigma_{i=1}^{k}\eta_{ni}f_i \geq t\}$$

$$\leq \lim_{t\to\infty} P\{\Sigma_{i=1}^{k}\eta_i f_i \geq t\} \text{ (Theorem 2.3.1 (iii))}$$

$$=0.$$

Thus $\{(\eta_{n1}, \ldots, \eta_{nk})\}_{n=1}^{\infty}$ is relatively compact by Lemma 2.3.3. Hence any subsequence $\mathbb{N}'$ of $\mathbb{N} = \{1, 2, 3, \ldots\}$ must contain a further subsequence $\mathbb{N}''$ such that $(\eta_{n1}, \ldots, \eta_{nk}) \xrightarrow{d}$ some $(\zeta_1, \ldots, \zeta_k)$, $n \in \mathbb{N}''$. Therefore, by Lemma 2.3.4,

$$\Sigma_{i=1}^{k}\eta_{ni}f_i \xrightarrow{d} \Sigma_1^k\zeta_i f_i,\ n \in \mathbb{N}'',\ f_i \in \mathcal{F}_c(S_i),\ i = 1, 2, \ldots, k.$$

Comparing this with (ii), we conclude that

$$\Sigma_1^k\eta_i f_i = \Sigma_1^k\zeta_i f_i,\ f_i \in \mathcal{F}_c(S_i),\ i = 1, \ldots, k.$$

By (ii) of Theorem 2.2.2, this implies that $(\eta_1, \ldots, \eta_k) \overset{d}{=} (\zeta_1, \ldots, \zeta_k)$, and thus we have $(\eta_{n1}, \ldots, \eta_{nk}) \overset{d}{\to} (\eta_1, \ldots, \eta_k)$, $n \in \mathbb{N}''$. This proves (i) by [5] Theorem 2.3.

Now suppose that (iii) holds. Then one may argue as above to show that for any given subsequence $\mathbb{N}'$ of $\mathbb{N}$, there exists a further subsequence $\mathbb{N}''$ such that

$$(\eta_{n1}, \ldots, \eta_{nk}) \overset{d}{\to} \text{some } (\zeta_1, \ldots, \zeta_k),\ n \in \mathbb{N}''.$$

However, one can not claim directly that

$$(2.3.3) \qquad (\Sigma_{i=1}^{k}\eta_{ni}(B_{i1}), \ldots, \Sigma_{i=1}^{k}\eta_{ni}(B_{im})) \overset{d}{\to}$$
$$(\Sigma_{i=1}^{k}\zeta_i(B_{i1}), \ldots, \Sigma_{i=1}^{k}\zeta_i(B_{im})),\ n \in \mathbb{N}'',$$
$$m = 1, 2, \ldots, B_{ij} \in \mathcal{T}_i,\ i = 1, \ldots, k,\ j = 1, \ldots, m.$$

This problem can be resolved by showing that $\mathcal{T}_i \subset \mathcal{B}_{\zeta_i}$, $i = 1, 2, \ldots, k$, in the following way. For each $i = 1, 2, \ldots, k$, let $_i$ be the ring generated by $\mathcal{T}_i$ and note that (iii) implies $\eta_{ni}(U) \overset{d}{\to} \eta_i(U)$, $U \in \mathcal{T}_i$ by Lemma 2.3.4 since U can be written as a finite union of disjoint members in $\mathcal{T}_i$. Hence $\mathcal{B}_{\zeta_i} \supset \mathcal{B}_{\eta_i} \supset \mathcal{T}_i$, $i = 1, 2, \ldots, k$ by [14], Lemma 4.6. Thus (2.3.3) holds by Lemma 2.3.4, and we have

$$(\eta_1, \ldots, \eta_k) \overset{d}{=} (\zeta_1, \ldots, \zeta_k)$$

by [14], Lemma 1.2 and (iii) of Theorem 2.2.4. Therefore $(\eta_{n1}, \ldots, \eta_{nk}) \overset{d}{\to} (\eta_1, \ldots, \eta_k)$, $n \in \mathbb{N}''$, which proves (i) by [5], Theorem 2.3. Q. E. D.

<u>Lemma 2.3.6</u> Let $\eta_1, \eta_2, \ldots$ be point processes on S and let $\mathcal{T} \subset \mathcal{B}(S)$ be a DC-semiring. Assume that for each $\varepsilon > 0$ and $U \in \mathcal{T}$, there exists a bounded set B such that ∂U is in the interior of B and $\limsup P\{\eta_n(B) > 0\} < \varepsilon$. Then the following holds:

(i) Suppose for each $k = 1, 2, \ldots$ and disjoint sets $U_1, U_2, \ldots, U_k$ in $\mathcal{T}$, $(\eta_n(U_1), \ldots, \eta_n(U_k))$ converges in distribution to some random element $\eta_{U_1 \ldots U_k}$ in I_+^k. Then η_n converges in distribution to some point process η.

(ii) If η_n converges in distribution to some point process η, then $\mathcal{T} \subset \mathcal{B}_\eta \overset{\text{def}}{=} \{B \in \mathcal{B}(S): \eta(\partial B) = 0 \text{ a.s.}\}$. In particular, this implies that $(\eta_n(U_1), \ldots, \eta_n(U_k)) \xrightarrow{d} (\eta(U_1), \ldots, \eta(U_k))$ for each $k = 1, 2, \ldots$ and sets $U_1, U_2, \ldots, U_k$.

<u>Proof</u>: We first prove (i). Since for each bounded set B, there exists a finite cover consisting of $\mathcal{T}$-sets $U_1, U_2, \ldots, U_k$,

$$\lim_{t\to\infty} \lim_{n\to\infty} P\{\eta_n(B) > t\} \le \lim_{t\to\infty} \lim_{n\to\infty} \Sigma_{i=1}^k P\{\eta_n(U_i) > t\} = 0.$$

This implies that $\{\eta_n\}$ is relatively compact by Lemma 2.3.3. Thus, for each subsequence $\mathbb{N}'$ of $\mathbb{N} = \{1, 2, 3, \ldots\}$, there exists a further subsequence $\mathbb{N}''$ such that $\eta_n \xrightarrow{d}$ some η, $n \in \mathbb{N}''$. Given any set U in $\mathcal{T}$ and constant $\varepsilon > 0$, there exists by assumption a bounded set B such that ∂U is in the interior of B and $\limsup P\{\eta_n(B) > 0\} < \varepsilon$. Since $\mathcal{B}_\eta$ contains a topological base (cf. [14], Lemma 4.3), there exists a set $c \in \mathcal{B}_\eta$ such that $\partial U \subset C \subset B$. By Lemma 2.3.4, $\eta_n(C)$ converges in distribution to $\eta(C)$, $n \in \mathbb{N}''$. The above facts and Theorem 2.3.1 now imply

$$P\{\eta(\partial U) > 0\} \le P\{\eta(C) > 0\} \le \liminf_{n \in \mathbb{N}''} P\{\eta_n(C) > 0\}$$

$$\le \limsup_{n \in \mathbb{N}} P\{\eta_n(B) > 0\} < \varepsilon,$$

showing that $\eta(\partial U) = 0$ a.s.. Hence for $U_1, U_2, \ldots, U_k \in \mathcal{T}$,

$$(\eta_n(U_1), \ldots, \eta_n(U_k)) \xrightarrow{d} (\eta(U_1), \ldots, \eta(U_k)), \; n \in \mathbb{N}'',$$

by Lemma 2.3.4. The assumption

$$(\eta_n(U_1), \ldots, \eta_n(U_k)) \xrightarrow{d} \eta_{U_1 \ldots U_k}, \; n \in \mathbb{N},$$

thus implies that $(\eta(U_1), \ldots, \eta(U_k)) \stackrel{d}{=} \eta_{U_1 \ldots U_k}$, $U_1, \ldots, U_k \in \mathcal{T}$. By Theorem 2.2.2 (iii), η is uniquely determined by the family $\{\eta_{U_1 \ldots U_k}, k = 1, 2, \ldots, U_1, \ldots, U_k \in \mathcal{T}\}$ and is therefore independent of the choice of $\mathbb{N}'$ and $\mathbb{N}''$. Thus we conclude $\eta_n \xrightarrow{d} \eta$, $n \in \mathbb{N}$, proving (i). The proof of (ii) is similar except that one could work with the limit η directly. Q. E. D.

CHAPTER III

THE CONDITION $\Delta(u_n)$ AND THE EXCEEDANCE POINT PROCESS ON [0, 1]

3.1 Introduction

Motivated by the results studied under the condition $D(u_n)$, we will introduce a mixing condition under which the limiting behavior of $N_n^{(\tau)}$ will be studied. It will be seen that the only possible limit laws of $N_n^{(\tau)}$ are Compound Poisson.

First of all, we define $N_n^{(\tau)}$ with the notation of Chapter II. For $\tau \in (0, \infty)$ and n = 1, 2, .., define

$$N_n^{(\tau)} = \Sigma_{j=1}^n \chi_{n,j}^{(\tau)} \delta_{j/n} \tag{3.1.1}$$

where

$$\chi_{n,j}^{(\tau)} = \begin{cases} 1 & \text{if } \xi_j > u_n^{(\tau)}, \\ 0 & \xi_j \le u_n^{(\tau)} \end{cases}$$

and δ_x, $x \in [0, 1]$ is the measure on [0, 1] with a unit mass at x. $N_n^{(\tau)}$ is a point process on [0, 1]. Note that the definition of $u_n^{(\tau)}$ only requires that

$$1 - F(u_n^{(\tau)}) \sim \tau/n. \tag{3.1.2}$$

There are apparently many such sequences and therefore the corresponding point processes $N_n^{(\tau)}$ are all different. Suppose now $\{u_{n,1}^{(\tau)}\}$ and $\{u_{n,2}^{(\tau)}\}$ are

two different sequences satisfying (3.1.2), and $N_{n,1}^{(\tau)}$ and $N_{n,2}^{(\tau)}$ are the corresponding point processes defined by (3.1.1). Then

$$P(N_{n,1}^{(\tau)} \neq N_{n,2}^{(\tau)}) \leq n|F(U_{n,1}^{(\tau)}) - F(u_{n,2}^{(\tau)})| \xrightarrow{n\to\infty} 0,$$

which implies that the distributional limits of $N_{n,1}^{(\tau)}$ and $N_{n,2}^{(\tau)}$ are the same provided that either one has a limit. Since we are only interested in convergence results, we therefore need not be specific about the choice of $\{u_n^{(\tau)}\}$ and indeed we can use any convenient $\{u_n^{(\tau)}\}$ to our advantage.

3.2 <u>The Mixing Condition $\Delta(u_n)$</u>

<u>Definition 3.2.1</u> Let $\{u_{n,m}\}_{n=1}^{\infty}$, m = 1, .., k, be k sequences of constants. For each n, i, j with $1 \leq i \leq j \leq n$, define $B_i^j(u_{n,1}, \ldots, u_{n,k}) = \sigma\{(\xi_s \leq u_{n,m}), i \leq s \leq j, 1 \leq m \leq k\}$. Also for each n and $1 \leq \ell \leq n - 1$, write

$$\alpha_{n,\ell} = \max\{|P(A \cap B) - P(A)\,P(B)|: A \in B_1^k(u_{n,1}, \ldots, u_{n,k}),$$

$$B \in B_{k+\ell}^n(u_{n,1}, \ldots, u_{n,k}), 1 \leq k \leq n - \ell\}.$$

$\{\xi_j\}$ is said to satisfy the <u>condition $\Delta(u_{n,1}, \ldots, u_{n,k})$</u> if $\alpha_{n,\ell_n} \to 0$ as $n \to \infty$ for some sequence $\{\ell_n\}$ with $\ell_n = o(n)$.

The array of constants $\alpha_{n,}$, $\ell = 1, 2, \ldots, n - 1$, will be referred to as the mixing coefficient of the condition $\Delta(u_{n,1}, \ldots, u_{n,k})$ whenever there is no danger of causing ambiguity.

It is worth nothing that the condition $\Delta(u_n)$ is stronger than the distributional mixing condition $D(u_n)$ but weaker than the strong mixing condition. For our purpose, $u_{n,i}$ will always be $u_n^{(\tau)}$ for some $\tau \in (0, \infty)$. Since there are only a finite number of events involved for each n, the condition $\Delta(u_n^{(\tau)})$ can be easily verified in some cases (cf. Chapter 5).

Indeed, the strong mixing condition is "unnecessarily strong" for most situations in the study of extreme value theory in that it poses restriction not just on the extremal but on the overall behavior of the underlying sequence. Finally, for the same reason as mentioned in section 1, the statement that the condition $\Delta(u_n^{(\tau)})$ holds for $\{\xi_j\}$ has the precise meaning that $\Delta(u_n^{(\tau)})$ holds for any sequence $\{u_n^{(\tau)}\}$ satisfying $1 - F(u_n^{(\tau)}) \sim \tau/n$.

The condition $\Delta(u_n)$ can be expressed in terms of random variables as well. The following result is a special case of [36], equation (I').

Lemma 3.2.2 For each n and $1 \leq \ell \leq n - 1$, write

$$\beta_{n,\ell} = \sup\{|E\eta\zeta - E\eta \cdot E\zeta| : \eta \text{ and } \zeta \text{ are measurable with respect to } \mathcal{B}_1^j(u_{n,1}, \ldots, u_{n,k}) \text{ and } \mathcal{B}_{j+\ell}^n(u_{n,1}, \ldots, u_{n,k}) \text{ respectively, } 0 \leq \eta, \zeta \leq 1,\ 1 \leq j \leq n - \ell\}.$$

Then $\alpha_{n,\ell} \leq \beta_{n,\ell} \leq 16\alpha_{n,\ell}$ where $\alpha_{n,\ell}$ is the mixing coefficient of the condition $\Delta(u_{n,1}, \ldots, u_{n,k})$. In particular, $\{\xi_j\}$ satisfies the condition $\Delta(u_{n,1}, \ldots, u_{n,k})$ if and only if $\beta_{n,\ell_n} \xrightarrow{n\to\infty} 0$ for some sequence $\{\ell_n\}$ with $\ell_n = o(n)$.

As noted in Chapter I, in order to study the limit of $N_n^{(\tau)}$, it is convenient to first divide $\xi_1, \xi_2, \ldots, \xi_n$ into groups. The appropriate size of the groups is given by the following definition.

Definition 3.2.3 Suppose $\{\xi_j\}$ satisfies the condition $\Delta(u_{n,1}, \ldots, u_{n,k})$. The sequence of positive integers $\{r_n\}_{n=1}^{\infty}$ is said to be $\Delta(u_{n,1}, \ldots, u_{n,k})$ -separating if $r_n/n \to 0$ and there exists a sequence $\{\ell_n\}$ such that $\ell_n/r_n \to 0$ and $n\alpha_{n,\ell_n}/r_n \to 0$, where $\alpha_{n,\ell}$, $\ell = 1, \ldots, n - 1$, are the mixing coeffi-

cients of $(\xi; u_{n,1}, \ldots, u_{n,k})$.

It is easy to see that such an r_n-sequence always exists and indeed one has considerable flexibility in choosing it. For example, if $\ell_n = o(n)$ is such that $\alpha_{n,\ell_n} \to 0$, then $\{r_n =$ the integer part of max $(n\alpha^{\frac{1}{2}}_{n,\ell_n}, (n\ell_n)^{\frac{1}{2}})\}$ is $\Delta(u_{n,1}, \ldots, u_{n,k})$-separating.

The following result demonstrates how the condition $\Delta(u_n)$ gives approximate independence of the number of exceedances in different groups.

<u>Lemma 3.2.4</u> Suppose $\tau_1, \tau_2, \ldots, \tau_k$ are positive constants and the condition $\Delta(u_n^{(\tau_1)}, \ldots, u_n^{(\tau_k)})$ holds for $\{\xi_j\}$. If $\{r_n\}$ is $\Delta(u_n^{(\tau_1)}, \ldots, u_n^{(\tau_k)})$-separating, then for $f_i \in F([0, 1])$, $i = 1, \ldots, k$, we have

(i) $$E\exp(-\Sigma_{m=1}^{k}\int_{[0,\ 1]} f_m dN_n^{(\tau_m)}) - \Pi_{i=1}^{[k_n]} E\exp(-\Sigma_{m=1}^{k}\int_{(i-1/k_n,\ i/k_n]} f_m dN_n^{(\tau_m)}) \xrightarrow{n\to\infty} 0,$$

(ii) $$E\exp(-\Sigma_{m=1}^{k}\int_{[0,\ 1]} f_m dN_n^{(\tau_m)}) \ / \ \Pi_{i=1}^{[k_n]} E\exp(-\Sigma_{m=1}^{k}\int_{(i-1/k_n,\ i/k_n]} f_m dN_n^{(\tau_m)}) \xrightarrow{n\to\infty} 1$$

where $k_n = n/r_n$ and $[x]$ = integer part of x.

<u>Proof</u>: We will show (i), (ii) follows from (i) and the fact

$$\liminf_{n\to\infty} E\exp(-\Sigma_{m=1}^{k}\int_{[0,\ 1]} f_m dN_n^{(\tau_m)}) \geq \liminf_{n\to\infty} P\{N_n^{(\tau)}([0, 1]) = 0\}$$

$$= \liminf_{n\to\infty} P\{M_n \leq u_n^{(\tau)}\} \geq e^{-\tau}$$

where $\tau = \max(\tau_i)$ (cf. [20], Theorem 2.2). Since $\{r_n\}$ is $\Delta(u_n^{(\tau_1)}, \ldots, u_n^{(\tau_k)})$-separating, there exists a sequence $\{\ell_n\}$ such that $\ell_n/n \to 0$ and $n\alpha_{n,\ell_n}/r_n \to 0$. For each n, write $I_{n,j} = \{(j-1)r_n + 1, (j-1)r_n + 2, \ldots, jr_n - \ell_n\}$,

$j = 1, \ldots, [k_n]$, and $I_n = \cup_{j=1}^{[k_n]} I_{n,j}$. By the triangle inequality,

$$|E\exp(-\Sigma_{m=1}^{k} \int_{[0,\ 1]} f_m dN_n^{(\tau_m)}) - \Pi_{i=1}^{[k_n]} E\exp(-\Sigma_{m=1}^{k} \int_{(i-1/k_n,\ i/k_n]} f_m dN_n^{(\tau_m)})|$$

$$= |E\exp(-\Sigma_{m=1}^{k} \Sigma_{j=1}^{n} f_m(j/n)\chi_{n,j}^{(\tau_m)}) - \Pi_{i=1}^{[k_n]} E\exp(-\Sigma_{m=1}^{k} \Sigma_{j=(i-1)r_n+1}^{ir_n} f_m(j/n)\chi_{n,j}^{(\tau_m)})|$$

(3.2.1)

$$\leq |E\exp(-\Sigma_{m=1}^{k} \Sigma_{j=1}^{n} f_m(j/n)\chi_{n,j}^{(\tau_m)}) - E\exp(-\Sigma_{m=1}^{k} \Sigma_{j\in I_n} f_m(j/n)\chi_{n,j}^{(\tau_m)})|$$

$$+ |E\exp(-\Sigma_{m=1}^{k} \Sigma_{j\in I_n} f_m(j/n)\chi_{n,j}^{(\tau_m)}) - \Pi_{i=1}^{[k_n]} E\exp(-\Sigma_{m=1}^{k} \Sigma_{j\in I_{n,i}} f_m(j/n)\chi_{n,j}^{(\tau_m)})|$$

$$+ |\Pi_{i=1}^{[k_n]} E\exp(-\Sigma_{m=1}^{k} \Sigma_{j\in I_{n,i}} f_m(j/n)\chi_{n,j}^{(\tau_m)}) - \Pi_{i=1}^{[k_n]} E\exp(-\Sigma_{m=1}^{k} \Sigma_{j=(i-1)r_n+1}^{ir_n} f_m(j/n)$$

$$\cdot \chi_{n,j}^{(\tau_m)})|$$

We will show that all three terms on the right hand side of (3.1.3) tends to zero. Since $f_1, f_2, \ldots, f_k$ are non-negative functions,

$$0 \leq E\exp(-\Sigma_{m=1}^{k} \Sigma_{j\in I_n} f_m(j/n)\chi_{n,j}^{(\tau_m)}) - E\exp(-\Sigma_{m=1}^{k} \Sigma_{j=1}^{n} f_m(j/n)\chi_{n,j}^{(\tau_m)})$$

$$= E\exp(-\Sigma_{m=1}^{k} \Sigma_{j\in I_n} f_m(j/n)\chi_{n,j}^{(\tau_m)}) [1 - \exp(-\Sigma_{m=1}^{k} \Sigma_{j\in\{1,\ldots,n\}-I_n} f_m(j/n)\chi_{n,j}^{(\tau_m)})]$$

$$\leq E[1 - \exp(-\Sigma_{m=1}^{k} \Sigma_{j\in\{1,\ldots,n\}-I_n} f_m(j/n)\chi_{n,j}^{(\tau_m)})]$$

$$\leq P\{\Sigma_{k=1}^{k} \Sigma_{j\in\{1,\ldots,n\}-I_n} f_m(j/n)\chi_{n,j}^{(\tau_m)} \neq 0\}$$

$$\leq P\{\ \xi_j > u_n^{(\tau)} \text{ for some } j \in \{1,\ldots,n\}-I_n\}\ (\tau = \max(\tau_i))$$

$$\leq ([k_n]\ell_n + r_n)\ [1 - F(u_n^{(\tau)})]$$

$$\sim (\ell_n/r_n + r_n/n)\ \tau \xrightarrow{n\to\infty} 0,$$

showing the first term on the right hand side of (3.2.1) tends to zero as n tend to ∞. To deal with the second term, note that

$$|E\exp(-\Sigma_{m=1}^{k}\Sigma_{j\in I_n} f_m(j/n)X_{n,j}^{(\tau_m)}) - \Pi_{i=1}^{[k_n]}E\exp(-\Sigma_{m=1}^{k}\Sigma_{j\in I_{n,i}} f_m(j/n)X_{n,j}^{(\tau_m)})|$$

$$\leq |E\exp(-\Sigma_{m=1}^{k}\Sigma_{j\in I_n} f_m(j/n)X_{n,j}^{(\tau_m)}) - E\exp(-\Sigma_{m=1}^{k}\Sigma_{j\in I_{n,1}} f_m(j/n)X_{n,j}^{(\tau_m)})$$

$$\times\ E\exp(-\Sigma_{m=1}^{k}\Sigma_{i=2}^{[k_n]}\Sigma_{j\in I_{n,i}} f_m(j/n)X_{n,j}^{(\tau_m)})| + E\exp(-\Sigma_{m=1}^{k}\Sigma_{j\in I_{n,1}} f_m(j/n)X_{n,j}^{(\tau_m)})$$

$$\times\ |E\exp(-\Sigma_{m=1}^{k}\Sigma_{i=2}^{[k_n]}\Sigma_{j\in I_{n,i}} f_m(j/n)X_{n,j}^{(\tau_m)}) - \Pi_{i=2}^{[k_n]}E\exp(-\Sigma_{m=1}^{k}\Sigma_{j\in I_{n,i}} f_m(j/n)\cdot$$

$$X_{n,j}^{(\tau_m)})|$$

$$\leq 16\alpha_{n,\ell_n} + |E\exp(-\Sigma_{m=1}^{k}\Sigma_{i=2}^{[k_n]}\Sigma_{j\in I_{n,i}} f_m(j/n)X_{n,j}^{(\tau_m)}) - \Pi_{i=2}^{[k_n]}E\exp(-\Sigma_{m=1}^{k}\Sigma_{j\in I_{n,i}}\cdot$$

$$f_m(j/n)X_{n,j}^{(\tau_m)})|$$

by Lemma 3.2.2 and the non-negativity of the f_i's. By induction,

$$|E\exp(-\Sigma_{m=1}^{k}\Sigma_{j\in I_n} f_m(j/n)X_{n,j}^{(\tau_m)}) - \Pi_{i=1}^{[k_n]}E\exp(-\Sigma_{m=1}^{k}\Sigma_{j\in I_{n,i}} f_m(j/n)X_{n,j}^{(\tau_m)})|$$

$$\leq 16\ [k_n]\ \alpha_{n,\ell_n} \xrightarrow{n\to\infty} 0.$$

the third term can be dealt with using the inequality

$$(3.2.2)\qquad |\Pi_{i=1}^{k}y_i - \Pi_{i=1}^{k}x_i| \leq \Sigma_{i=1}^{k}|y_i - x_i|,\ 0 \leq y_i,\ x_i \leq 1,\ i = 1, \ldots, k,$$

showing that

$$|\Pi_{i=1}^{[k_n]}E\exp(-\Sigma_{m=1}^{k}\Sigma_{j\in I_{n,i}} f_m(j/n)X_{n,j}^{(\tau_m)}) - \Pi_{i=1}^{[k_n]}E\exp(-\Sigma_{m=1}^{k}\Sigma_{j=(i-1)r_n+1}^{ir_n} f_m(j/n)\cdot$$

$$X_{n,j}^{(\tau_m)})|$$

$$\leq \Sigma_{i=1}^{[k_n]}|E\exp(-\Sigma_{m=1}^{k}\Sigma_{j\in I_{n,i}} f_m(j/n)X_{n,j}^{(\tau_m)}) - E\exp(-\Sigma_{m=1}^{k}\Sigma_{j=(i-1)r_n+1} f_m(j/n)\cdot$$

$$X_{n,j}^{(\tau_m)})$$

$$\leq [k_n]\ell_n[1 - F(u_n^{(\tau)})] \xrightarrow{n\to\infty} 0.$$

This concludes the proof. Q. E. D.

3.3 Compound Poisson Convergence of $N_n^{(\tau)}$

Now we state one of the main results of this section.

Theorem 3.3.1 Suppose $\infty > \tau_1 > \tau_2 > .. > \tau_k > 0$ are constants and the condition $\Delta(u_n^{(\tau)}, .., u_n^{(\tau_k)})$ holds for $\{\xi_j\}$. If $(N_n^{(\tau_1)}, .., N_n^{(\tau_k)})$ converges in distribution to some $(N^{(\tau_1)}, .., N^{(\tau_k)})$, then the latter must have a Laplace Transform of the form

$$\exp(-\int_0^1(1 - L(f_1(t), .., f_k(t)))\theta\tau_1 dt),$$

where $L(s_1, .., s_k)$ is the Laplace Transform of some probability measure π on $I_k \overset{\text{def}}{=} \{(i_1, .., i_k) \in I_+^k: i_1 \geq 1 \text{ and } i_1 \geq i_2 \geq .. \geq i_k\}$ and $\theta = -1/\tau_1 \lim_{n\to\infty} \log P\{M_n \leq u_n^{(\tau_1)}\} \in [0, 1]$. If $\theta \neq 0$, then for each $\Delta(u_n^{(\tau_1)}, .., u_n^{(\tau_k)})$-separating $\{r_n\}$,

$$\pi(\{\underset{\sim}{i}\}) = \lim_{n\to\infty} P\{\Sigma_{j=1}^{r_n} X_{n,j}^{(\tau_m)} = i_m, m = 1, .., k|\Sigma_{j=1}^{r_n} X_{n,j}^{(\tau_1)} > 0\},$$

$\underset{\sim}{i} = (i_1, .., i_k) \in I_k$.

The implication of the theorem is most obvious when $k = 1$. In this

$(i_1, \ldots, i_k) \in \mathcal{I}_k$.

Here we ignored the trivial case $\theta = 0$, which leads to $N^{(\tau)}(\mathbb{R}) = 0$ a.s. for each $\tau > 0$. When there is no danger of ambiguity, $(N^{(\tau_1)}, \ldots, N^{(\tau_k)})$ always denotes the distributional limit of $(N_n^{(\tau_1)}, \ldots, N_n^{(\tau_k)})$, $\tau_1, \tau_2, \ldots, \tau_k > 0$.

4.3 Asymptotic Distribution of kth Largest Values

We now apply our convergence results to problems that are of concern of the more traditional theory. Let $M_n^{(k)}$ be the k-th largest among $\xi_1, \xi_2, \ldots, \xi_n$. It is easy to see that $(M_n^{(k)} \le u_n^{(\tau)})$ is the same event as $(N_n^{(\tau)} \le k-1)$. Using this fact, one can derive limiting distributions for properly normalized $M_n^{(k)}$.

Theorem 4.3.1 Suppose that for each $\tau > 0$, $\Delta(u_n^{(\tau)})$ holds for $\{\xi_j\}$ and $N_n^{(\tau)}$ converges in distribution to some non-trivial point process $N^{(\tau)}$. Assume that $a_n > 0$, b_n are constants such that

$$P\{a_n(M_n - b_n) \le x\} \xrightarrow{w} G(x)$$

for some non-degenerate distribution function G. Then for each k=1, 2, 3, ...,

$$\lim_{n\to\infty} P\{a_n(M_n^{(k)} - b_n) \le x\}$$
$$= \Sigma_{j=0}^{k-1} G(x) \frac{[-\log G(x)]^j}{j!} \pi^{*j}(\{1, 2, \ldots, k-1\})$$

(where $G(x) > 0$, and zero where $G(x) = 0$), where π^{*j} is the j-fold convolution of the probability measure defined by

$$\pi(\{i\}) = \lim_{n\to\infty} P\{\Sigma_{j=1}^{r_n} X_{n,j}^{(\tau)} = i \mid \Sigma_{j=1}^{r_n} X_{n,j}^{(\tau)} > 0\}$$

$$= \exp\{-\theta\tau_1 \int_{\mathbb{R}} (1 - L(g_1(t), \ldots, g_k(t)))dt\}.$$

Thus by (4.2.3), (4.2.4) and a change of variable,

$$(4.2.5) \quad \lim_{n\to\infty} E\exp(-\Sigma_{m=1}^{k} \int_{\mathbb{R}} f_m dN_n^{(\sigma\tau_m)}) = \exp\{-\theta\sigma\tau_1 \int_{\mathbb{R}} (1-L(f_1(t),\ldots,f_k(t)))dt\}.$$

it follows from Theorem 2.3.5 that $(N_n^{(\sigma\tau_1)}, \ldots, N_n^{(\sigma\tau_k)})$ converges in distribution to some $(N^{(\sigma\tau_1)}, \ldots, N^{(\sigma\tau_k)})$ whose Laplace Transform is given by (4.2.5), and it is now obvious that θ is the extremal index. Q. E. D.

We remark that under the assumptions of the above result, the distribution π which determines the cluster sizes of the limit Compound Poisson processes depends on $\tau_1, \ldots, \tau_k$ only through $\tau_2/\tau_1, \ldots, \tau_k/\tau_1$ if $k > 1$, and is independent of τ if $k = 1$.

The following result can be proved in a similar way using Theorem 3.3.4 and Lemma 4.2.2.

<u>Theorem 4.2.4</u> Let $\tau_1 \geq \tau_2 \geq \ldots \geq \tau_k > 0$ be constants. Suppose $\{\xi_j\}$ satisfies the condition $\Delta(u_n^{(\sigma\tau_1)}, \ldots, u_n^{(\sigma\tau_k)})$ for each $\sigma > 0$. Also assume that $\{\xi_j\}$ has extremal index $\theta \in (0, 1]$, and there exists a $\Delta(u_n^{(\tau_1)}, \ldots, u_n^{(\tau_k)})$-separating $\{r_n\}$ such that for each $\underset{\sim}{i} \in I_k$, $P\{\Sigma_{j=1}^{r_n} \chi_{n,j}^{(\tau_m)} = i_m$, $m = 1, \ldots, k \,|\, \Sigma_{j=1}^{r_n} \chi_{n,j}^{(\tau_1)} > 0\}$ converges as $n \to \infty$. Then for each $\sigma > 0$, $(N_n^{(\sigma\tau_1)}, \ldots, N_n^{(\sigma\tau_k)})$ converges in distribution to some point process $(N^{(\sigma\tau_1)}, \ldots, N^{(\sigma\tau_k)})$ with Laplace Transform $\exp[-\theta\sigma\tau_1 \int_{\mathbb{R}} (1-L(f_1(t), \ldots, f_k(t)))dt]$ where L is the Laplace Transform of the probability measure π on I_k determined by

$$\pi(\{(i_1, \ldots, i_k)\}) = \lim_{n\to\infty} P\{\Sigma_{j=1}^{r_n} \chi_{n,j}^{(\tau_m)} = i_m,\ m=1,\ldots,k \,|\, \Sigma_{j=1}^{r_n} \chi_{n,j}^{(\tau_1)} > 0\},$$

$$+ \left| E\exp(-\Sigma_{m=1}^{k}\Sigma_{0\le j/n\le v} g_m(j/[n/\sigma])\chi_{[n/\sigma],j}^{(\tau_m)}) - E\exp(-\Sigma_{m=1}^{k}\Sigma_{0\le j/[n/\sigma]\le v\sigma} g_m(j/[n/\sigma])\chi_{[n/\sigma],j}^{(\tau_m)})\right|$$

It follows from the inequality (4.2.1) that the above expression is bounded by

$$\Sigma_{m=1}^{k}\Sigma_{0\le j/n\le v}|F(u_n^{(\sigma\tau_m)})-F(u_{[n/\sigma]}^{(\tau_m)})|+\Sigma_{m=1}^{k}\Sigma_{0\le j/n\le v}|e^{-f_m(j/n)}-e^{-g_m(j/[n/\sigma])}|\cdot$$

$$(1 - F(u_{[n/\sigma]}^{(\tau)})) + \Sigma_{m=1}^{k}\Sigma_{\sigma[n/\sigma]\le j\le v\sigma}(1 - F(u_{[n/\sigma]}^{(\tau_m)}))$$

where all three terms tend to zero by the definition of $u_n^{(\tau)}$ and the choice of the f's. Thus

$$(4.2.3)\quad \lim_{n\to\infty} E\exp(-\Sigma_{m=1}^{k}\int_{\mathbb{R}} f_m dN_n^{(\sigma\tau_m)}) = \lim_{n\to\infty} E\exp(-\Sigma_{m=1}^{k}\int_{\mathbb{R}} g_m dN_n^{(\tau_m)})$$

provided the latter limit exists, which is true by the assumption that $(N_n^{(\tau_1)}, \ldots, N_n^{(\tau_k)})$ converges in distribution. Since the limits $N^{(\tau_1)}, \ldots, N^{(\tau_k)}$ do not have fixed atoms, Lemma 2.3.4 implies that

$$E\exp(-\Sigma_{m=1}^{k}\int_{(0,1]} g_m dN^{(\tau_m)}) = \lim_{n\to\infty} E\exp(-\Sigma_{m=1}^{k}\int_{(0,1]} g_m dN_n^{(\tau_m)})$$

and it follows by arguments similar to those in Theorem 3.3.1 that the last expression equals $\exp\{-\theta\tau_1\int_0^1(1 - L(g_1(t), \ldots, g_k(t)))dt\}$ where L is as stated in the theorem and θ is such that $\lim_{n\to\infty} P\{M_n^{(1)} \le u_n^{(\tau_1)}\} = e^{-\theta\tau_1}$. Note that the supports of $g_1, \ldots, g_k$ are in $(0, \sigma v]$, Lemma 4.2.2 thus implies that

$$\lim_{n\to\infty} E\exp(-\Sigma_{m=1}^{k}\int_{\mathbb{R}} g_m dN_n^{(\tau_m)}) = \Pi_{i=1}^{[\sigma v]+1} \lim_{n\to\infty} E\exp(-\Sigma_{m=1}^{k}\int_{(i-1,i]} g_m dN_n^{(\tau_m)})$$

$$(4.2.4)\quad = \Pi_{i=1}^{[\sigma v]+1}\exp\{-\theta\tau_1\int_{(i-1,i]}(1 - L(g_1(t), \ldots, g_k(t)))dt\}$$

Lemma 3.2.4 since the condition $\Delta(u_n^{(v\tau_1)}, .., u_n^{(v\tau_m)})$ holds for $\{\xi_j\}$. The conclusin follows. Q. E. D.

Theorem 4.2.3 Suppose $\infty > \tau_1 \geq \tau_2 \geq .. \geq \tau_k > 0$ are constants, and the condition $\Delta(u_n^{(\sigma\tau_1)}, .., u_n^{(\sigma\tau_k)})$ holds for $\{\xi_j\}$ for each $\sigma > 0$. If $(N_n^{(\tau_1)}, .., N_n^{(\tau_k)})$ converges in distribution to some $(N^{(\tau_1)}, .., N^{(\tau_k)})$, then for each $\sigma > 0$, $(N_n^{(\sigma\tau_1)}, .., N_n^{(\sigma\tau_k)})$ converges in distribution to some $(N^{(\sigma\tau_1)}, .., N^{(\sigma\tau_k)})$ with Laplace Transform

$$\exp\{-\theta\sigma\tau_1\int_{\mathbb{R}}(1 - L(f_1(t), .., f_k(t)))dt\}$$

where $\vartheta \in [0, 1]$ is the extremal index, which exists, and L is the Laplace Transform of some probability measure π on I_k. If $\theta \neq 0$, π is determined by

$$\pi\{(i_1, .., i_k)\} = \lim_{n\to\infty} P\{\Sigma_{j=1}^{r_n}\chi_{n,j}^{(\tau_m)} = i_m,\ m = 1, 2, .., k \,|\, \Sigma_{j=1}^{r_n}\chi_{n,j}^{(\tau_1)} > 0\}$$

where $\{r_n\}$ is any $\Delta(u_n^{(\tau_1)}, .., u_n^{(\tau_k)})$-separating sequence.

Proof: Let $\sigma > 0$ be fixed, and $f_1, .., f_k$ be functions in $F_c(\mathbb{R})$ with supports in $(0, v]$ for some $0 < v < \infty$. Write $g_m(t) = f_m(t/\sigma)$, $m = 1,2,..,k$. By the triangle inequality,

$$|E\exp(-\Sigma_{m=1}^{k}\int_{\mathbb{R}} f_m dN_n^{(\sigma\tau_m)}) - E\exp(-\Sigma_{m=1}^{k}\int_{\mathbb{R}} g_m dN_{[n/\sigma]}^{(\tau_m)})|$$

$$= |E\exp(-\Sigma_{m=1}^{k}\Sigma_{0\leq j/n\leq v} f_m(j/n)\chi_{n,j}^{(\sigma\tau_m)}) - E\exp(-\Sigma_{m=1}^{k}\Sigma_{0\leq j/[n/\sigma]\leq v\sigma} g_m(j/[n/\sigma])\cdot \chi_{[n/\sigma],j}^{(\tau_m)})|$$

$$\leq |E\exp(-\Sigma_{m=1}^{k}\Sigma_{0\leq j/n\leq v} f_m(j/n)\chi_{n,j}^{(\sigma\tau_m)}) - E\exp(-\Sigma_{m=1}^{k}\Sigma_{0\leq j/n\leq v} f_m(j/n)\chi_{[n/\sigma],j}^{(\tau_m)})|$$

$$+ |E\exp(-\Sigma_{m=1}^{k}\Sigma_{0\leq j/n\leq v} f_m(j/n)\chi_{[n/\sigma],j}^{(\tau_m)}) - E\exp(-\Sigma_{m=1}^{k}\Sigma_{0\leq j/n\leq v} g_m(j/[n/\sigma])\cdot \chi_{[n/\sigma],j}^{(\tau_m)})|$$

The results in Section 3.3 can now be extended as follows.

Lemma 4.2.2 Let $\tau_1, \tau_2, \ldots, \tau_k$ be positive constants. Assume that $\{\xi_j\}$ satisfies the condition $\Delta(u_n^{(\sigma\tau_1)}, \ldots, u_n^{(\sigma\tau_k)})$ for each $\sigma > 0$. Then for functions $f_1, f_2, \ldots, f_k$ in $F(\mathbb{R})$ with supports in some bounded interval $(u - 1, v]$, u, v being integers, we have

$$E\exp(-\Sigma_{m=1}^{k}\int_{\mathbb{R}} f_m dN_n^{(\tau_m)}) - \Pi_{i=u}^{v} E\exp(-\Sigma_{m=1}^{k}\int_{(i-1,\ i]} f_m dN_n^{(\tau_m)})$$

$\to 0$ as $n \to \infty$.

Proof: Since $\{\xi_j\}$ is stationary, we can assume without loss of generality that $u = 1$. Hence

$$E\exp(-\Sigma_{m=1}^{k}\int_{\mathbb{R}} f_m dN_n^{(\tau_m)}) = E\exp(-\Sigma_{m=1}^{k}\Sigma_{j=1}^{vn} f_m(j/n)\chi_{n,j}^{(\tau_m)})$$

By the triangle inequality,

$$|E\exp(-\Sigma_{m=1}^{k}\Sigma_{j=1}^{vn} f_m(j/n)\chi_{n,j}^{(\tau_m)}) - \Pi_{i=1}^{v} E\exp(-\Sigma_{m=1}^{k}\Sigma_{j=(i-1)n+1}^{in} f_m(j/n)\chi_{n,j}^{(\tau_m)})|$$

$$\leq |E\exp(-\Sigma_{m=1}^{k}\Sigma_{j=1}^{vn} f_m(j/n)\chi_{n,j}^{(\tau_m)}) - E\exp(-\Sigma_{m=1}^{k}\Sigma_{j=1}^{vn} f_m(j/n)\chi_{vn,j}^{(v\tau_m)})|$$

$$+ |E\exp(-\Sigma_{m=1}^{k}\Sigma_{j=1}^{vn} f_m(j/n)\chi_{vn,j}^{(v\tau_m)}) - \Pi_{i=1}^{v} E\exp(-\Sigma_{m=1}^{k}\Sigma_{j=(i-1)n+1}^{in} f_m(j/n)\chi_{vn,j}^{(v\tau_m)})|$$

$$+ |\Pi_{i=1}^{v} E\exp(-\Sigma_{m=1}^{k}\Sigma_{j=(i-1)n+1}^{in} f_m(j/n)\chi_{vn,j}^{(v\tau_m)}) - \Pi_{i=1}^{v} E\exp(-\Sigma_{m=1}^{k}\Sigma_{j=(i-1)n+1}^{in} f_m (j/n)\chi_{n,j}^{(\tau_m)})|.$$

The first and third term tend to zero by the obvious inequality

$$|\Pi_{i=1}^{k} x_i - \Pi_{i=1}^{k} y_i| \leq \Sigma_{i=1}^{k}|x_i - y_i|,\ 0 \leq x_i,\ y_i \leq 1,\ i = 1, \ldots, k,$$

and the fact $n|F(u_n^{(\tau)}) - F(u_{vn}^{(v\tau)})| \to 0$. The second term tends to zero by

and each choice of $\tau_1, \tau_2, .., \tau_k > 0$. For convenience, call the above assumption <u>the Δ condition</u>. Again, the condition Δ is weaker than strong mixing.

4.2 Point Processes of Exceedance Positions on $\mathbb{R}$

In Chapter III, we restricted $N_n^{(\tau)}$ to be a point process on $[0, 1]$. We shall see that such a restriction makes little sense under the more stringent mixing condition Δ. Instead we consider the point process $N_n^{(\tau)}$ on $\mathbb{R}$ defined by

$$N_n^{(\tau)} = \Sigma_{j\in I}\, \chi_{n,j}^{(\tau)}\, \delta_{j/n}, \quad \tau > 0, \ n \in \mathbb{N},$$

where $\chi_{n,j}^{(\tau)} = 1(\xi_j > u_n^{(\tau)})$, $j \in I$, and δ_a, $a \in \mathbb{R}$, is the measure with a unit mass at a. We commence by stating a result which is slightly more general than what is needed for the present purpose.

<u>Lemma 4.2.1</u> Let $\tau > 0$ be a constant. If $N_n^{(\tau)}$ converges in distribution to some point process N, then N does not have fixed atoms; i.e., $N(\{s\}) = 0$ a.s. for each $s \in \mathbb{R}$.

<u>Proof</u>: Since $\mathcal{B}_n$ contains a topological base (cf. [14], Lemma 4.3), for each $s \in \mathbb{R}$ and $\varepsilon > 0$, there exists a set $B \in \mathcal{B}_n$ such that $s \in B \subset (s - \varepsilon, s + \varepsilon)$. Thus, by Theorem 2.3.1,

$$\begin{aligned} P\{N(\{s\}) > 0\} &\le P\{N(B) > 0\} \le \liminf_n P\{N_n^{(\tau)}(B) > 0\} \\ &\le \liminf_n P\{N_n^{(\tau)}((s-\varepsilon, s+\varepsilon)) > 0\} \le \lim_n (2n\varepsilon+1)\, P\{\xi_1 > u_n^{(\tau)}\} \\ &= 2\varepsilon\tau. \end{aligned}$$

This concludes the claim since ε is arbitrary. Q. E. D.

CHAPTER IV

COMPLETE CONVERGENCE

4.1 Introduction

Let $\{\xi_j\}_I$ be a stationary sequence with marginal distribution F. Recall that $\{u_n^{(\tau)}\}$ is a sequence for which $n[1 - F(u_n^{(\tau)})] \to \tau$ as $n \to \infty$. For simplicity, we now require, in addition, that $u_n^{(\tau)}$ be strictly decreasing in τ for each n so that u_n^{-1} is well defined. For example, suppose F belongs to the domain of attraction of some extreme value type distribution G, and constants $a_n > 0$ and b_n are such that $F^n(a_n x + b_n) \to G(x)$. Then $u_n^{(\tau)}$ can be taken to be $a_n^{-1}G^{-1}(e^{-\tau}) + b_n$. While the restriction is not essential, the removal of it would cause extra complexity and would not add depth to the general theory. Write $\mathbb{N}_n$ for the point process $\Sigma_{j\in I}\, \delta_{(j/n,\, u_n^{-1}(\xi_j))}$ where $\delta_{(a,\, b)}$, $(a, b) \in \mathbb{R} \times \mathbb{R}_+' = (-\infty, \infty) \times (0, \infty)$, is the measure on $\mathbb{R} \times \mathbb{R}_+'$ with a unit mass at (a, b). $\mathbb{N}_n$ is a point process on $\mathbb{R} \times \mathbb{R}_+'$. As before, the convergence results for $\mathbb{N}_n$ are not affected by the choice of $\{u_n^{(\tau)}\}_{n,\tau}$. Mori [26] shows, under a slightly different setting, that if $\{\xi_j\}$ is strongly mixing, the limit $\mathbb{N}$ (in distribution) of $\mathbb{N}_n$, when it exists, must be infinitely divisible and invariant under certain transformations. Using these facts, he further characterizes $\mathbb{N}$ in terms of its canonical measure. We propose to both give sufficient conditions for the convergence of $\mathbb{N}_n$ and characterize the limit $\mathbb{N}$ under the assumption that $\Delta(u_n^{(\tau_1)}, \ldots, u_n^{(\tau_k)})$ holds for $\{\xi_j\}$ for each k = 1, 2, ..

$$\tilde{\pi}(\{(i_1, \ldots, i_k)\}) = \lim_{n \in \mathbb{N}_2} \pi_n(\{(i_1, \ldots, i_k)\}).$$

This implies that $\tilde{\pi} = \pi$ and the assertion follows. Q. E. D.

The uninteresting case $\theta = 0$ was left out in the above theorem. In this case,

$$\lim_{n \to \infty} P\{N_n^{(\tau)}([0, 1]) = 0\} = \lim_{n \to \infty} P\{M_n \leq u_n^{(\tau)}\} = 1,$$

showing that the limit of $N_n^{(\tau)}$ equals the null measure almost surely by Theorem 2.3.1.

$\{r_n\}$ such that for each $\underset{\sim}{i} \in I_k$, $P\{\Sigma_{j=1}^{r_n} X_{n,j}^{(\tau_m)} = i_m,\ m=1,\ 2,\ \ldots,\ k \mid \Sigma_{j=1}^{r_n} X_{n,j}^{(\tau_1)} > 0\}$ converges as $n \to \infty$. Then

(i) the measure π on I_k defined by

$$\pi(\{\underset{\sim}{i}\}) = \lim_{\to\infty} P\{\Sigma_{j=1}^{r_n} X_{n,j}^{(\tau_m)} = i_m,\ m = 1,2,\ldots,k \mid \Sigma_{j=1}^{r_n} X_{n,j}^{(\tau_1)} > 0\},$$

$\underset{\sim}{i} = (i_1, \ldots, i_k) \in I_k$, is a probability measure;

(ii) $(N_n^{(\tau_1)}, \ldots, N_n^{(\tau_k)})$ converges in distribution to some $(N^{(\tau_1)}, \ldots, N^{(\tau_k)})$ with Laplace Transform

$$\exp\{-\theta\tau_1 \int_0^1 [1 - L(f_1(t), \ldots, f_k(t))]dt\}$$

where L is the Laplace Transform of π.

Proof: We will only show (i), (ii) follows by arguments similar to those in Theorem 3.3.1. Let π_n be the probability measure on I_k defined by

$$\pi_n(\{(i_1, \ldots, i_k)\}) = P\{\Sigma_{j=1}^{r_n} X_{n,j}^{(\tau_m)} = i_m,\ m = 1,\ \ldots,\ k \mid \Sigma_{j=1}^{r_n} X_{n,j}^{(\tau_1)} > 0\}$$

$(i_1, \ldots, i_k) \in I_k$. Write Q_n for the probability measure on $\mathbb{N}$ defined by

$$Q_n(\{i\}) = P\{\Sigma_{j=1}^{r_n} X_{n,j}^{(\tau_1)} = i \mid \Sigma_{j=1}^{r_n} X_{n,j}^{(\tau_1)} > 0\} = \Sigma_{i_2=0}^{i} \Sigma_{i_3=0}^{i_2} \cdots \Sigma_{i_k=0}^{i_{k-1}} \pi_n(\{(i, i_2, \ldots, i_k)\})$$

$i \in \mathbb{N}$. It is clear that $\{\pi_n\}$ is tight if $\{Q_n\}$ is, which follows readily from

$$\lim_{n\to\infty} \Sigma_{k=1}^{\infty} k\, Q_n(\{k\}) = 1/\theta < \infty \text{ (cf. (1.5.4))}.$$

Therefore, by Prohorov's Theorem, for any infinite subset $\mathbb{N}_1$ of $\mathbb{N}$, there exists a further subset $\mathbb{N}_2$ and a probability measure $\tilde{\pi}$ on I_k such that $\pi_n \Rightarrow \pi$, $n \in \mathbb{N}_2$. Thus for each $(i_1, \ldots, i_k) \in I_k$,

By (3.3.1), (3.3.2) and (3.3.3),

$$\log E\exp(-\Sigma_{m=1}^{k}\int_{[0,\ 1]} f_m dN_n^{(\tau_m)})$$

$$= -k_n\int_0^1 R_n(t)dt - k_n\int_0^1 \psi(R_n(t))dt + o(1)$$

$$= -\theta\tau_1\int_0^1 [1 - \Sigma\exp(-\Sigma_{m=1}^{k} f_m(t)j_m)\pi_n(\{\underset{\sim}{j}\})]dt + o(1),$$

which converges as $n \to \infty$ by the assumption that $(N_n^{(\tau_1)}, \ldots, N_n^{(\tau_k)})$ converges and Theorem 2.3.5. But this implies that the limit $\lim_{n\to\infty} \Sigma\exp(-s_m j_m)\cdot \pi_n(\{\underset{\sim}{j}\})$ exists for each $(s_1, \ldots, s_k) \in \mathbb{R}_+^k$, which is equivalent to the existence of a measure π on I_k such that $\pi(\{\underset{\sim}{j}\}) = \lim_{n\to\infty} \pi_n(\{\underset{\sim}{j}\})$ for each $\underset{\sim}{j} \in I_k$, and in this case,

$$\lim_{n\to\infty} \Sigma\exp(-\Sigma_{m=1}^{k} s_m j_m)\pi_n(\{\underset{\sim}{j}\}) = \Sigma\exp(-\Sigma_{m=1}^{k} s_m j_m)\pi(\{\underset{\sim}{j}\})$$

for each $(s_1, \ldots, s_k) \in \mathbb{R}_+^k$. We conclude, by dominated convergence, that $\lim_{n\to\infty} \log E\exp(-\Sigma_{m=1}^{k}\int_{[0,\ 1]} f_m dN_n^{(\tau_m)}) = -\theta\tau_1\int_0^1 [1 - \Sigma\exp(-\Sigma_{m=1}^{k} f_m(t)j_m)\pi(\{\underset{\sim}{j}\})]dt.$ Theorem 2.3.5 now implies that

$$E\exp(-\Sigma_{m=1}^{k}\int_{[0,1]} f_m dN^{(\tau_m)}) = \exp\{-\theta\tau_1\int_0^1 [1-\Sigma\exp(-\Sigma_{m=1}^{k} f_m(t)j_m)\pi(\{\underset{\sim}{j}\})]dt$$

where π is seen to be a probability measure since, for example, $N^{(\tau_1)}, \ldots, N^{(\tau_k)}$ are point processes. Q. E. D.

One can also state a constructive result as follows.

<u>Theorem 3.3.4</u> Suppose $\{\xi_j\}$ satisfies the condition $\Delta(u_n^{(\tau_1)}, \ldots, u_n^{(\tau_k)})$ for constants $\infty > \tau_1 > \tau_2 > \ldots > \tau_k > 0$. Assume that $\lim_{n\to\infty} P\{M_n \le u_n^{(\tau_1)}\} = e^{-\theta\tau_1}$ for some $\theta \in (0, 1]$ and that there exists a $\Delta(u_n^{(\tau_1)}, \ldots, u_n^{(\tau_k)})$-separating

$$1_{(0,\ [k_n]/k_n)}(t)$$

$$= (\theta\tau_1+o(1))[1 - \Sigma_{\underset{\sim}{j}\in I_k}\exp(-\Sigma_{m=1}^{k}f_m(t)j_m)\pi_n(\{\underset{\sim}{j}\})]1_{(0,\ [k_n]/k_n)}(t)$$

Proving (ii). Q. E. D.

We now prove Theorem 3.3.1.

<u>Proof of Theorem 3.3.1</u>: Lemma 3.3.2 concludes that there exists a $\theta \in [0, 1]$ such that $P\{M_n \leq u_n^{(\tau)}\} \xrightarrow{n\to\infty} e^{-\theta\tau}$ for each $\tau \in (0, \tau_1]$. If $\theta = 0$, then the conclusion follows immediately. Suppose that $\theta \neq 0$. Let R_n and $\tilde{R}_n$ be as defined in Lemma 3.3.3. By Lemma 3.2.4,

$$\log E\exp(-\Sigma_{m=1}^{k}\int_{[0,\ 1]}f_m dN_n^{(\tau_m)})$$

$$= \Sigma_{i=1}^{[k_n]}\log E\exp(-\Sigma_{m=1}^{k}\int_{(i-1/k_n,\ i/k_n]}f_m dN_n^{(\tau_m)}) + o(1)$$

(3.3.1)

$$= k_n\Sigma_{i=1}^{[k_n]}1/k_n\log[1-(1-E\exp(-\Sigma_{m=1}^{k}\int_{(i-1/k_n,\ i/k_n]}f_m dN_n^{(\tau_m)}))] + o(1)$$

$$= k_n\int_0^1\log[1 - R_n(t)]dt + o(1).$$

Lemma 3.3.3 implies that

(3.3.2) $$k_nR_n(t)-\theta\tau_1[1-\Sigma_{\underset{\sim}{j}\in I_k}\exp(-\Sigma_{m=1}^{k}f_m(t)j_m)\pi_n(\{\underset{\sim}{j}\})]1_{(0,\ [k_n]/k_n)}(t) \to 0$$

uniformly in t. Let $\psi(x) = -\log(1 - x) - x$, $x \in [0, 1)$, $\psi(x) \sim x^2/2$ as $x \to 0$. Hence for large n, $|\psi(R_n(t))| \leq R_n^2(t)$ for all t since $R_n(t) \to 0$ uniformly in t. As a consequence,

(3.3.3) $$k_n\int_0^1|\psi(R_n(t))|dt \leq 1/k_n\int_0^1[k_nR_n(t)]^2dt \xrightarrow{n\to\infty} 0.$$

$$|k_n(R_n(t) - \tilde{R}_n(t))|$$

$$\leq k_n \Sigma_{i=1}^{[k_n]} \Sigma_{m=1}^{k} \Sigma_{j/n \in (i-1/k_n,\ i/k_n]} |e^{-f_m(j/n)} - e^{-f_m(t)}| P\{\xi_1 > u_n^{(\tau_m)}\}.$$

$$1_{(i-1/k_n,\ i/k_n]}(t)$$

$$\leq k_n r_n \Omega_n \Sigma_{i=1}^{[k_n]} 1_{(i-1/k_n,\ i/k_n]}(t) \Sigma_{m=1}^{k} P\{\xi_1 > u_n^{(\tau_m)}\}$$

where $\Omega_n = \sup\{|e^{-f_m(s)} - e^{-f_m(t)}| : |s - t| < 1/k_n,\ m = 1, 2, \ldots, k\}$. It is easily seen that $\Omega_n \xrightarrow{n\to\infty} 0$ since $1/k_n \xrightarrow{n\to\infty} 0$ and $f_1, f_2, \ldots, f_k$ are uniformly continuous. Thus

$$|k_n(R_n(t) - \tilde{R}_n(t))|$$

$$\leq k_n r_n \Omega_n\ \Sigma_{m=1}^{k} P\{\xi_1 > u_n^{(\tau_m)}\}$$

$$\sim k_n r_n \Omega_n\ \Sigma_{m=1}^{k} \tau_m/n$$

$$= \Omega_n \Sigma_{m=1}^{k} \tau_m \xrightarrow{n\to\infty} 0,$$

showing (i). To show (ii), first note that

$$P\{\Sigma_{j=1}^{r_n} \chi_{n,j}^{(\tau_1)} > 0\} = P\{M_{r_n} > u_n^{(\tau_1)}\} \sim \theta\tau_1/k_n$$

by Lemma 1.5.3. Hence by stationarity,

$$k_n \tilde{R}_n(t) = k_n \Sigma_{s=1}^{[k_n]} [1 - P\{\Sigma_{i=1}^{r_n} \chi_{n,s}^{(\tau_1)} = 0\} - \Sigma_{\underset{\sim}{j} \in I_k} \exp(-\Sigma_{m=1}^{k} f_m(t) j_m) \pi_n(\{\underset{\sim}{j}\}) \cdot$$

$$P\{\Sigma_{s=1}^{r_n} \chi_{n,s}^{(\tau_1)} > 0\}] 1_{(i-1/k_n,\ i/k_n]}(t)$$

$$= k_n P\{\Sigma_{s=1}^{r_n} \chi_{n,s}^{(\tau_1)} > 0\} [1 - \Sigma_{\underset{\sim}{j} \in I_k} \exp(-\Sigma_{m=1}^{k} f_m(t) j_m) \pi_n(\{\underset{\sim}{j}\})] \cdot$$

$$R_n(t) = \Sigma_{i=1}^{[k_n]}[1-E\exp(-\Sigma_{m=1}^{k}\int_{(i-1/k_n,\ i/k_n]} f_m dN_n^{(\tau_m)})]1_{(i-1/k_n,\ i/k_n]}(t),$$

$$\tilde{R}_n(t) = \Sigma_{i=1}^{[k_n]}[1-E\exp(-\Sigma_{m=1}^{k} f_m(t)N_n^{(\tau_m)}((i-1/k_n,\ i/k_n]))]1_{(i-1/k_n,\ i/k_n]}(t)$$

where $\{r_n\}$ is $\Delta(u_n^{(\tau_1)}, \ldots, u_n^{(\tau_k)})$-separating, $k_n = n/r_n$ and $f_1, \ldots, f_k$ are continuous functions on $[0, 1]$. Then as $n \to \infty$,

(i) $k_n(R_n(t) - \tilde{R}_n(t)) \to 0$ uniformly in t,

(ii) $k_n\tilde{R}_n(t)-\theta\tau_1[1-\Sigma_{\underset{\sim}{j}\in I_k}\exp(-\Sigma_{m=1}^{k} j_m f_m(t))\pi_n(\{\underset{\sim}{j}\})\ 1_{(0,\ [k_n]/k_n]}(t) \to 0$

uniformly in t where $\pi_n(\{\underset{\sim}{j}\})=P\{\Sigma_{i=1}^{r_n}\chi_{n,i}^{(\tau_m)}=j_m,\ m=1, \ldots, k \mid \Sigma_{i=1}^{r_n}\chi_{n,j}^{(\tau_1)} > 0\}$,

$\underset{\sim}{j} = (j_1, \ldots, j_k) \in I_k$.

<u>Proof</u>: $\quad |k_n(R_n(t) - \tilde{R}_n(t))|$

$$\le k_n\Sigma_{i=1}^{[k_n]}|E\exp(-\Sigma_{m=1}^{k}\int_{(i-1/k_n,\ i/k_n]} f_m dN_n^{(\tau_m)})-E\exp(-\Sigma_{m=1}^{k} f_m(t)\cdot$$

$$N_n^{(\tau_m)}((i-1/k_n,\ i/k_n]))|1_{(i-1/k_n,\ i/k_n]}(t)$$

$$= k_n\Sigma_{i=1}^{[k_n]}|E\exp(-\Sigma_{m=1}^{k}\Sigma_{j/n\in(i-1/k_n,\ i/k_n]} f_m(j/n)\chi_{n,j}^{(\tau_m)})-E\exp(-\Sigma_{m=1}^{k} f_m(t)\cdot$$

$$\Sigma_{j/n\ (i-1/k_n,\ i/k_n]}\chi_{n,j}^{(\tau_m)})|1_{(i-1/k_n,\ i/k_n]}(t)$$

$$\le k_n\Sigma_{i=1}^{[k_n]}\Sigma_{m=1}^{k}\Sigma_{j/n\in(i-1/k_n,\ i/k_n]}|E\exp(-f_m(j/n)\chi_{n,j}^{(\tau_m)})-E\exp(-f_m(t)\cdot$$

$$\chi_{n,j}^{(\tau_m)})|1_{(i-1/k_n,\ i/k_n]}(t)$$

by (3.2.2). Since $|E\exp(-f_m(j/n)\chi_{n,j}^{(\tau_m)})-E\exp(-f_m(t)\chi_{n,j}^{(\tau_m)})|=|e^{-f_m(j/n)}-e^{-f_m(t)}|\cdot$ $P\{\xi_1>u_n^{(\tau_m)}\}$, we have

case, if $N_n^{(\tau)} \xrightarrow{d} N^{(\tau)}$, then the Laplace Transform of $N^{(\tau)}$ is $\exp(-\theta\tau\int_0^1 (1 - L(f(t)))dt)$, showing that $N^{(\tau)}$ is Compound Poisson. When $\theta \neq 0$, the probability measure π that corresponds to L is obviously restricted to a certain class; for example, by Fatou's Lemma,

$$\begin{aligned}\Sigma_{i=1}^{\infty} i\pi(\{i\}) &= \Sigma_{i=1}^{\infty} i \lim_{n\to\infty} P\{\Sigma_{j=1}^{r_n} X_{n,j}^{(\tau)} = i \mid \Sigma_{j=1}^{r_n} X_{n,j}^{(\tau)} > 0\} \\ &\leq \liminf_{n\to\infty} \Sigma_{i=1}^{\infty} iP\{\Sigma_{j=1}^{r_n} X_{n,j}^{(\tau)} = i \mid \Sigma_{j=1}^{r_n} X_{n,j}^{(\tau)} > 0\} \\ &= \lim_{n\to\infty} E(\Sigma_{j=1}^{r_n} X_{n,j}^{(\tau)} \mid \Sigma_{j=1}^{r_n} X_{n,j}^{(\tau)} > 0) \\ &= 1/\theta\end{aligned}$$

(cf. (1.4.4)) where $\{r_n\}$ is $\Delta(u_n^{(\tau)})$-separating. The precise relationship between θ and π is still an open problem.

We first prove two lemmas.

Lemma 3.3.2 Suppose $\{\xi_j\}$ satisfies the condition $\Delta(u_n^{(\tau)})$ for some $\tau > 0$, and $N_n^{(\tau)}$ converges in distribution. Then there exists a $\theta \in [0, 1]$ such that $\lim_{\to\infty} P\{M_n \leq u_n^{(\delta)}\} = e^{-\theta\delta}$ for each $\delta \in (0, \tau]$.

Proof: Since [0, 1] is a bounded set with empty boundary, the assumption impliesthat $N_n^{(\tau)}([0, 1])$ converges in distribution (cf. Lemma 2.3.4). Thus, the conclusion follows from the identity $P\{M_n \leq u_n^{(\tau)}\} = P\{N_n^{(\tau)}([0, 1]) = 0\}$ and Theorem 1.4.2. Q. E. D.

Lemma 3.3.3 Let $\tau_1 > \tau_2 > .. > \tau_k > 0$ constants. Suppose the condition $\Delta(u_n^{(\tau_1)}, ..., u_n^{(\tau_k)})$ holds for $\{\xi_j\}$ and there exists a $\theta \in [0, 1]$ for which $\lim_{n\to\infty} P\{M_n \leq u_n^{(\tau_1)}\} = e^{-\theta\tau_1}$. Define two functions $R_n(t)$ and $\tilde{R}_n(t)$ on [0, 1] by

$i = 1, 2, 3, \ldots$, for any arbitrary $\tau > 0$ and arbitrary sequence $\{r_n\}$ which is $\Delta(u_n^{(\tau)})$-separating.

Proof: According to Theorem 4.2.3, the Laplace Transform of $N^{(\tau)}$ is

$$\exp(-\theta\tau\int_{\mathbb{R}}[1 - L(f(t))]dt) \tag{4.3.1}$$

where $\theta \in (0, 1]$ is the extremal index of $\{\xi_j\}$ and L the Laplace Transform of the probability measure π stated in the theorem, L and π being independent of the choice of the positive constant τ and the $\Delta(u_n^{(\tau)})$-separating sequence $\{r_n\}$ by the remark following Theorem 4.2.3. The fact that $\{\xi_j\}$ has a non-zero extremal index θ and $P\{a_n(M_n - b_n) \leq x\} \xrightarrow{w} G(x)$ imply (cf. [20], Theorem 2.5) that G is one of the three extreme value type distributions, and

$$\lim_{n\to\infty} P\{a_n(\hat{M}_n - b_n) \leq x\} = G^{1/\theta}(x)$$

where $\hat{M}_n$ is the maximum of n independent random variables all having the same distributions as ξ_1. Thus

$$\lim_{n\to\infty} P\{\hat{M}_n \leq a_n^{-1}G^{-1}(e^{-\theta\tau}) + b_n\}$$

$$= G^{1/\theta}(G^{-1}(e^{-\theta\tau})) = e^{-\tau},$$

which shows by Proposition 1.2.1 that

$$1 - F(a_n^{-1}G^{-1}(e^{-\theta\tau}) + b_n) \sim \tau/n \text{ as } n \to \infty.$$

Writing $\tau(x) = -\log G^{1/\theta}(x)$, we get

$$1 - F(a_n^{-1}x + b_n) \sim \tau(x)/n. \tag{4.3.2}$$

Now it follows from (4.3.1), (4.3.2) and the fact $N_n^{(\tau)}((0,1]) \xrightarrow{d} N^{(\tau)}((0,1])$

(cf. Lemma 4.2.1 and Lemma 2.3.4) that

$$\lim_{n\to\infty} P\{a_n(M_n^{(k)} - b_n) \le x\}$$

$$= \lim_{n\to\infty} P\{M_n^{(k)} \le u_n^{(\tau(x))}\}$$

$$= \lim_{n\to\infty} P\{N_n^{(\tau(x))}((0, 1]) \le k-1\}$$

$$= P\{N^{(\tau(x))}((0, 1]) \le k-1\}$$

$$= \Sigma_{j=0}^{k-1} e^{-\theta\tau(x)} \frac{(\theta\tau(x))^j}{j!} \pi^{*j}(\{1, 2, .., k-1\})$$

$$= \Sigma_{j=0}^{k-1} G(x) \frac{[-\log G(x)]^j}{j!} \pi^{*j}(\{1, 2, .., k-1\}). \qquad \text{Q. E. D.}$$

Using the same idea, the asymptotic joint distribution of a finite number of the k-th largest maxima $M_n^{(k)}$ can be obtained.

<u>Theorem 4.3.2</u> Suppose that for each $\tau_1, \tau_2 > 0$, $\Delta(u_n^{(\tau_1)}, u_n^{(\tau_2)})$ holds for $\{\xi_j\}$ and $(N_n^{(\tau_1)}, N_n^{(\tau_2)})$ converges in distribution to some non-trivial $(N^{(\tau_1)}, N^{(\tau_2)})$. Assume that $a_n > 0$, b_n are constants such that

$$P\{a_n(M_n - b_n) \le x\} \overset{w}{\to} G(x)$$

for some non-degenerate distribution function G (which is one of the three extreme types by [21], Theorem 3.3.3). Then

$$\lim_{n\to\infty} P\{a_n(M_m^{(1)} - b_n) \le x,\ a_n(M_n^{(2)} - b_n) \le y\}$$

$$= \begin{cases} G(y)[1 - \rho(\frac{\log G(x)}{\text{lob}G(y)}) \cdot \log G(y)] & y < x,\ G(y) > 0, \\ G(x) & y \ge x,\ G(x) > 0, \\ 0 & \text{otherwise}, \end{cases}$$

where ρ, a function on $(0, 1)$, is defined by

$$\rho(\sigma) = \lim_{n\to\infty} P\{\Sigma_{j=1}^{r_n} \chi_{n.j}^{(\tau)} = 1,\ \Sigma_{j=1}^{r_n} \chi_{n,j}^{(\sigma\tau)} = 0 \mid \Sigma_{j=1}^{r_n} \chi_{n,j}^{(\tau)} > 0\}$$

for any arbitrary $\tau > 0$ and $\Delta(u_n^{(\tau)}, u_n^{(\sigma\tau)})$-separating $\{r_n\}$.

Proof: We only prove the assertion for the non-trivial situation $y < x$, $G(y) > 0$. By Theorem 4.2.3, the Laplace Transform of $(N^{(\tau)}, N^{(\sigma\tau)})$, $\tau > 0$, $\sigma \in (0,1)$, is

$$\exp(-\theta\tau \int_{\mathbb{R}} [1 - L(f_1(t), f_2(t))]dt)$$

where L is the Laplace Transform of the probability measure π on I_2 satisfying

$$\pi(\{(i_1, i_2)\}) = \lim_{n\to\infty} P\{\Sigma_{j=1}^{r_n} \chi_{n,j}^{(\tau)} = i_1,\ \Sigma_{j=1}^{r_n} \chi_{n,j}^{(\sigma\tau)} = i_2 \mid \Sigma_{j=1}^{r_n} \chi_{n,j}^{(\tau)} > 0\},$$

$\{r_n\}$ being any $\Delta(u_n^{(\tau)}, u_n^{(\sigma\tau)})$-separating sequence. By the comment that follows Theorem 4.2.3, π depends on $(\tau, \sigma\tau)$ only through the ratio σ. Thus, write $\pi(\cdot; \sigma)$ for π to emphasize this dependence.

Suppose $y < x$, $G(y) > 0$. By the notation and arguments used in Theorem 4.3.1,

$$\begin{aligned}
&\lim_{n\to\infty} P\{a_n(M_n^{(1)} - b_n) \leq x,\ a_n(M_n^{(2)} - b_n) \leq y\} \\
&= \lim_{n\to\infty} P\{M_n^{(1)} \leq u_n^{(\tau(x))},\ M_n^{(2)} \leq u_n^{(\tau(y))}\} \\
&= \lim_{n\to\infty} (P\{N_n^{(\tau(y))}((0, 1]) = 1,\ N_n^{(\tau(x))}((0, 1]) = 0\} + P\{N_n^{(\tau(y))}((0, 1]) = 0\}) \\
&= \theta\tau(y)e^{-\theta\tau(y)}\pi(\{(1, 0)\}; \tau(x)/\tau(y)) + e^{-\theta\tau(y)} \\
&= e^{-\theta\tau(y)}[1 + \theta\tau(y)\pi(\{(1, 0)\}; \tau(x)/\tau(y))] \\
&= G(y)[1 - \pi(\{(1, 0)\}; \log G(x)/\log G(y))\log G(y)].
\end{aligned}$$

Writing $\rho(\sigma) = \pi(\{(1, 0)\}; \sigma)$, $\sigma \in (0, 1)$, the result follows. Q. E. D.

We now state an interesting result due to Welsch ([37]).

Theorem 4.3.3 Let $\{X_n\}$ be a stationary strong-mixing sequence. If there exists a sequence of constants $\{a_n > 0, b_n: n \geq 1\}$ so that $P\{M_n^{(1)} \leq a_n x + b_n, M_n^{(2)} \leq a_n y + b_n\}$ has a limiting distribution, $H(x, y)$, with $G(x)$, the limiting distribution of $P\{M_n \leq a_n x + b_n\}$ non-degenerate, then

$$H(x, y) = \begin{cases} G(y)[1 - \rho(\log G(x)/\log G(y))\log G(y)] & y < x \\ G(x) & y \geq x \end{cases}$$

where $\rho(s)$, $0 \leq s \leq 1$ is a concave, non-increasing function which satisfies $\rho(0)(1 - s) \leq \rho(s) \leq 1 - s$. G is one of the three extreme types and we interpret $(\infty/\infty) = 1$, $(0/0) = 1$ and $(0/\infty) = 0$.

As can be seen easily, Theorem 4.3.2 is not an attempt to improve Welsch's result. However, it properly explains the role of ρ from the point of view of exceedance.

4.4 The Convergence of $\mathbb{N}_n$

In Chapter III, the method of Laplace Transform was used to show convergence of point processes. We now demonstrate another useful technique.

Let $\mathcal{P} = \{[a, b) \times [c, d): -\infty < a < b < \infty, 0 < c < d < \infty\}$. $\mathcal{P}$ is obviously a DC-semiring contained in $\mathcal{B}(\mathbb{R} \times \mathbb{R}_+')$.

Lemma 4.4.1 Suppose for each $k = 1, 2, 3, ..$ and $U_1, ..., U_k \in \mathcal{P}$, $(\mathbb{N}_n(U_1), ..., \mathbb{N}_n(U_k))$ converges in distribution. Then $\mathbb{N}_n$ converges in distribution to some point process $\mathbb{N}$. Suppose, on the other hand, that

$\mathbb{N}_n$ converges in distribution to some point process $\mathbb{N}$, Then $P \subset \mathcal{B}_{\mathbb{N}} = \{B \in \mathcal{B}(\mathbb{R} \times \mathbb{R}_+'): \mathbb{N}(\partial B) = 0 \text{ a.s.}\}$.

Proof: The boundary of a set in P contains at most four finite lines, each of which is of the form $[a, b] \times \{c\}$, $-\infty < a < b < \infty$, $c > 0$, or the form $\{u\} \times [v, w]$, $0 < v < w < \infty$, $u \in \mathbb{R}$. By Lemma 2.3.6, it suffices to show that for each ε and each line L of the above forms, there exists a bounded set B in $\mathbb{R} \times \mathbb{R}_+$ such that L is in the interior of B and $\limsup P\{\mathbb{N}_n(B) > 0\} < \varepsilon$. Consider, for example, $L = [a, b] \times \{c\}$. Choose $0 < \delta < \min(\varepsilon/2(b-a+2c), c)$. L is contained in the interior of $[a-\delta, b+\delta] \times [c-\delta, c+\delta)$, and

$$P\{\mathbb{N}_n([a-\delta, b+\delta] \times [c-\delta, c+\delta)) > 0\}$$

$$= P\{\Sigma_{j/n \in [a-\delta, a+\delta]} (X_{n,j}^{(c+\delta)} - X_{n,j}^{(c-\delta)}) > 0\}$$

$$\le ((b - a + 2\delta)n + 1)\, P\{u_n^{(c+\delta)} < \xi_1 \le u_n^{(c-\delta)}\}$$

$$\to 2(b - a + 2\delta)\delta < \varepsilon.$$

The other form of L can be dealt with similarly, proving the lemma.

It is obvious that there is a close relationship between the convergence of $(N_n^{(\tau_1)}, \ldots, N_n^{(\tau_k)})$ and that of $\mathbb{N}_n$. The following result makes the relationship precise. Like the previous result, it does not depend on any mixing assumption.

Theorem 4.4.2 $\mathbb{N}_n \overset{d}{\to}$ some point process $\mathbb{N}$ if and only if $(N_n^{(\tau_1)}, N_n^{(\tau_2)}, \ldots, N_n^{(\tau_k)}) \overset{d}{\to} (N^{(\tau_1)}, N^{(\tau_2)}, \ldots, N^{(\tau_k)})$ for each $k = 1, 2, \ldots$ and $\tau_1, \tau_2, \ldots, \tau_k > 0$. In this case,

$$(4.4.1) \qquad (\mathbb{N}(\cdot\times(0,\ \tau_1)),\ \ldots,\ \mathbb{N}(\cdot\times(0,\ \tau_k))) \overset{d}{=} (N^{(\tau_1)},\ \ldots,\ N^{(\tau_k)})$$

for each choice of $\tau_1, \ldots, \tau_k > 0$.

<u>Proof</u>: First assume that $(N_n^{(\tau_1)}, \ldots, N_n^{(\tau_k)})$ converges in distribution for each $k = 1, 2, 3, \ldots$ and $\tau_1, \tau_2, \ldots, \tau_k > 0$. For $U_i = [a_i, b_i) \times [c_i, d_i) \in P$, $i = 1, \ldots, m$,

$$(\mathbb{N}_n(U_1),\ \ldots,\ \mathbb{N}_n(U_m))$$

$$\overset{d}{=} (N_n^{(d_1)}([a_1, b_1))-N_n^{(c_1)}([a_1, b_1)),\ \ldots,\ N_n^{(d_m)}([a_m, b_m))-N_n^{(c_m)}([a_m, b_m)))$$

$$\overset{d}{\to} (N^{(d_1)}([a_1, b_1))-N^{(c_1)}([a_1, b_1)),\ \ldots,\ N^{(d_m)}([a_m, b_m))-N^{(c_m)}([a_m, b_m)))$$

by Lemma 4.2.1 and Lemma 2.3.4. Thus, by Lemma 4.4.1, N_n converges in distribution to some point process N with

$$(\mathbb{N}(U_1),\ \ldots,\ \mathbb{N}(U_m))$$

$$\overset{d}{=} (N^{(d_1)}([a_1, b_1))-N^{(c_1)}([a_1, b_1)),\ \ldots,\ N^{(d_m)}([a_m, b_m))-N^{(c_m)}([a_m, b_m)))$$

for each choice of $U_i = [a_i, b_i) \times [c_i, d_i) \in P$, $i = 1, \ldots, m$. In particular, it can be shown simply that

$$(\mathbb{N}([a_1, b_1)\times(0,\ \tau_1)),\ \ldots,\ \mathbb{N}([a_k, b_k)\times(0,\ \tau_k)))$$

$$\overset{d}{=} (N^{(\tau_1)}([a_1, b_1)),\ \ldots,\ N^{(\tau_k)}([a_k, b_k)))$$

for each choice of $\tau_1, \tau_2, \ldots, \tau_k > 0$ and $[a_1, b_1), \ldots, [a_k, b_k)$. It now follows from Theorem 2.2.2 (iii) that

$$(\mathbb{N}(\cdot\times(0,\ \tau_1)),\ \ldots,\ \mathbb{N}(\cdot\times(0,\ \tau_k))) \overset{d}{=} (N^{(\tau_1)},\ \ldots,\ N^{(\tau_k)}).$$

The converse can be shown similarly. Q. E. D.

4.5 The Characterization of $\mathbb{N}$ under Δ

Suppose that the condition Δ holds for $\{\xi_j\}$ and $\mathbb{N}_n$ converges in distribution to some point process $\mathbb{N}$. By Theorem 4.4.2 and Theorem 4.2.3, the finite dimensional distributions of $\mathbb{N}$ can be derived from (4.4.1) and (4.2.4). While the distribution of $\mathbb{N}$ is determined by the finite dimensional distributions, this knowledge does not provide a clear picture of $\mathbb{N}$. It is desirable to transform the knowledge into a description which is more "visible", so to speak. To do so, we approach from the point of view of "infinite divisibility" —— a technique used by Mori [26].

Lemma 4.5.1 Assume that $\{\xi_j\}$ satisfies the condition Δ, and $\mathbb{N}_n$ converges in distribution to some point process $\mathbb{N}$. Then $\mathbb{N}$ is an infinitely divisible point process.

Proof: By [14], Lemma 6.3, it suffices to show that $(\mathbb{N}(U_1), \ldots, \mathbb{N}(U_k))$ is infinitely divisible for each choice of $U_i = [a_i, b_i) \times [c_i, d_i) \in \mathcal{P}$, $i = 1, 2, \ldots, k$. It is simply seen that

$$(\mathbb{N}(U_1), \ldots, \mathbb{N}(U_k))$$

$$= (\mathbb{N}([a_1, b_1)\times[0, d_1)) - \mathbb{N}(a_1, b_1)\times[0, c_1)), \ldots, \mathbb{N}([a_k, b_k)\times[0, d_k)) - \mathbb{N}([a_k, b_k)\times[0, c_k)))$$

$$\stackrel{d}{=} T(N^{(d_1)}([a_1, b_1)), N^{(c_1)}([a_1, b_1)), \ldots, N^{(d_k)}([a_k, b_k), N^{(c_k)}([a_k, b_k)))$$

by Theorem 4.4.2 where T is the linear map

$$T(x_1, y_1, \ldots, x_k, y_k) = (x_1-y_1, \ldots, x_k-y_k).$$

Therefore it suffices to show that $(N^{(\tau_1)}(B_1), \ldots, N^{(\tau_m)}(B_m))$ is infinitely

divisible for each choice of constants $\tau_1, \tau_2, \ldots, \tau_k > 0$ and Borel sets $B_1, B_2, \ldots, B_m$ in $\mathbb{R}$. For a fixed choice, let k be the number of different τ_i, and τ'_j, $j = 1, \ldots, k$, the j-th largest value of $\tau_1, \ldots, \tau_m$. By Theorem 4.2.3, the L. T. of $(N^{(\tau_1)}(B_1), \ldots, N^{(\tau_m)}(B_m))$ is of the form

$$E\exp(-\Sigma_{i=1}^{m} z_i N^{(\tau_i)}(B_i)) = E\exp(-\Sigma_{j=1}^{k} \int_{\mathbb{R}} f_j(t) N^{(\tau'_j)}(dt))$$

$$= \exp[-\theta\tau'_1 \int_{\mathbb{R}} (1 - L(f_1(t), \ldots, f_k(t)))dt],$$

where $f_j(t) = \Sigma z_i 1_{B_i}(t)$, the summation extending over the set of i's for which $\tau_i = \tau'_j$. Also, by Theorem 4.2.3, for each $k = 1, 2, 3, \ldots$,

$$E\exp(-\Sigma_{i=1}^{m} z_i N^{(\tau_i/k)}(B_i)) = \exp(-\frac{\theta\tau'_1}{k} \int_{\mathbb{R}} (1 - L(f_1(t), \ldots, f_k(t)))dt),$$

showing that $E\exp(-\Sigma_{i=1}^{m} z_i N^{(\tau_i)}(B_i)) = [E\exp(-\Sigma_{i=1}^{m} z_i N^{(\tau_i/k)}(B_i))]^k$. The conclusion follows. Q. E. D.

If $\mathbb{N}$ is infinitely divisible, write $\tilde{P}$ for the canonical measure of $\mathbb{N}$. $\tilde{P}$ is a measure on $N(\mathbb{R} \times \mathbb{R}'_+) \setminus \{o\}$, o being the null measure on $\mathbb{R} \times \mathbb{R}_+$ (cf. Theorem 2.2.2).

Write $\bar{N}([1, \infty))$ for the collection of all locally finite integer-valued measures ψ on $[1, \infty)$ such that $\psi\{1\} \geq 1$. As a space, $\bar{N}([1, \infty))$ is equipped with the vague topology and the Borel σ-field $\bar{\mathcal{N}}([1, \infty))$. In what follows, we shall consider mappings between $N(\mathbb{R} \times \mathbb{R}'_+) \setminus \{o\}$ and $\bar{N}([1, \infty))$. The measurability of the mappings can be established by routine arguments, and will not be pursued specifically. To describe mappings between spaces of measures, it is often convenient to consider the corresponding transformations for "atoms". To do so, we first let ε_y, $y \in [1, \infty)$ and δ_x, $x \in \mathbb{R} \times \mathbb{R}'_+$ be the Dirac measures on $[1, \infty)$ and $\mathbb{R} \times \mathbb{R}'_+$ respectively.

Now let g be a measurable mapping on $(\mathbb{R} \times \mathbb{R}_+') \times \overline{N}([1, \infty))$ into $(\mathbb{R} \times \mathbb{R}_+')\setminus\{o\}$ defined by $g(x, \psi) = \Sigma a_i \delta_{(s,\ ty_i)}$ where $x = (s, t) \in \mathbb{R} \times \mathbb{R}_+'$, $\psi \in \overline{N}([1, \infty))$ and has a decomposition $a_i \varepsilon_{y_i}$ (cf. Chapter 2). Since $(\mathbb{R} \times \mathbb{R}_+') \times \overline{N}([1, \infty))$ and $N(\mathbb{R} \times \mathbb{R}_+')\setminus\{o\}$ are both Polish, Kuratowski's Theorem (cf. [29]) implies that g maps measurable sets to measurable sets. Let Λ be the range of g, namely, $\Lambda = \{\phi \in N(\mathbb{R} \times \mathbb{R}_+')\setminus\{o\}: \phi = g(x, \psi),\ x \in \mathbb{R} \times (0, \infty),\ \psi \in \overline{N}([1, \infty))\}$.

<u>Lemma 4.5.2</u> Suppose the condition Δ holds for $\{\xi_j\}$ and $\mathbb{N}_n$ converges in distribution to some point process $\mathbb{N}$. Then $\tilde{P}$, the canonical measure of $\mathbb{N}$, concentrates on Λ.

Proof: Since $\tilde{P}$ is a measure on $N(\mathbb{R} \times \mathbb{R}_+')\setminus\{o\}$, it is to be understood that all set operations are performed in this space. First it is obvious that

$$(4.5.1)\quad \Lambda = \{\phi \in N(\mathbb{R} \times \mathbb{R}_+')\setminus\{o\}: \phi(\{s\} \times \mathbb{R}_+) = 0 \text{ for all but one } s \in \mathbb{R}\}$$

$$= \cap_{m=1}^{\infty} \cap_{n=1}^{\infty} A_{m,n}$$

where

$$A_{mn} = \{\phi \in N(\mathbb{R} \times \mathbb{R}_+')\setminus\{o\}: \phi([k/2^n, k+1/2^n) \times [0, m)) = 0$$

$$\text{for all but possibly one } k \text{ in } I\}.$$

Note that A_{mn} is monotonically non-increasing in m for each fixed n, and $\cap_{m=1}^{\infty} A_{mn}$ is also monotonically non-increasing in n. Thus

$$(4.5.2)\quad \tilde{P}\ \{\phi \in N(\mathbb{R} \times \mathbb{R}_+')\setminus\{o\}: \phi(\{s\} \times \mathbb{R}_+') = 0 \text{ for all but one } s \in \mathbb{R}\}^c$$

$$= \lim_{n\to\infty} \lim_{m\to\infty} \tilde{P}\,(A^c_{mn})$$

$$\le \lim_{n\to\infty} \lim_{m\to\infty} \Sigma_{i\neq j}\ \tilde{P}\,\{\phi \in N(\mathbb{R} \times \mathbb{R}'_+)\setminus\{o\}: \phi([i/2^n,\ i+1/2^n)\times[0,m)) > 0,$$

$$\phi([j/2^n,\ j+1/2^n) \times [0,\ m)) > 0\}$$

$$= 0$$

By Lemma 2.2.4 since

$$(\mathbb{N}([i/2^n,\ i+1/2^n) \times [0,\ m)),\ \mathbb{N}([j/2^n,\ j+1/2^n) \times [0,\ m)))$$

$$\stackrel{d}{=} (N^{(m)}([i/2^n,\ i+1/2^n)),\ N^{(m)}([j/2^n,\ j+1/2^n))),$$

$N^{(m)}([i/2^n,\ i+1/2^n))$ being independent of $N^{(m)}([j/2^n,\ j+1/2^n))$ if $i\neq j$ (cf. Theorem 4.4.2 and Theorem 4.2.3). Q. E. D.

For each $\tau \in \mathbb{R}$ and $\sigma > 0$, define mappings u_τ and v_σ by

$$u_\tau(s,\ t) = (s+\tau,\ t),$$

$$v_\sigma(s,\ t) = (\sigma s,\ t/\sigma),$$

$(s,\ t) \in \mathbb{R} \times \mathbb{R}'_+$.

<u>Lemma 4.5.3</u> A measure μ on $\mathbb{R} \times \mathbb{R}'_+$ is a scalor multiple of Lebesque measure if and only if $\mu \circ u_\tau = \mu \circ v_\sigma = \mu$ for all $\tau \in \mathbb{R}$ and $\sigma > 0$.

<u>Proof</u>: The "only if" part is trivial. To show the "if" Part", let $[a,\ b) \times [c,\ d) \subset \mathbb{R} \times \mathbb{R}'_+$, the assumption implies that

(4.5.4) $\mu([a,\ b) \times [c,\ d)) = \mu([0,\ b-a) \times [c,\ d)),$

(4.5.5) $\mu([a,\ b) \times [c,\ d)) = \mu([\sigma a,\ \sigma b) \times [c/\sigma,\ d/\sigma)),\ \sigma > 0.$

For each m = 1, 2, 3, ...,

$$\mu([0, 1) \times [0, 1))$$

$$= \Sigma_{k=1}^{m} \mu([k-1/m, k/m) \times [0, 1))$$

$$= m\mu([0, 1/m) \times [0, 1)),$$

by (4.5.4). Thus $\mu([0, 1/m) \times [0, 1)) = 1/m\ \mu([0, 1) \times [0, 1))$, which implies that $\mu([0, n/m) \times [0, 1)) = n/m\ \mu([0, 1) \times [0, 1))$ for each m, n= 1, 2, 3, ... Since the set of rationals is dense in $\mathbb{R}$, we have

(4.5.6) $$\mu([a, b) \times [0, 1))$$

$$= \mu([0, b-a) \times [0, 1))$$

$$= \lim_{n/m \uparrow (b-a)} \mu([0, n/m) \times [0, 1))$$

$$= (b-a)\ \mu\ ([0, 1) \times [0, 1)).$$

Let t > 0 be arbitrary,

(4.5.7) $$\mu([a, b) \times [0, t))$$

$$= \mu([0, b-a) \times [0, t))$$

$$= \mu([0, (b-a)t) \times [0, 1))$$

$$= (b-a)t \cdot \mu([0, 1) \times [0, 1)).$$

Hence $$\mu([a, b) \times [c, d))$$

$$= \mu([a, b) \times [0, d)) - \mu([a, b) \times [0, c))$$

$$= \{(b-a)d - (b-a)c\}\ \mu([0, 1) \times [0, 1)) \qquad \text{by (4.5.7)}$$

$= (b-a)\ (d-c)\ \mu\ ([0,\ 1) \times [0,\ 1)).$

The conclusion follows since $\{[a,b)\times[c,d)\ -\infty < a < b < \infty,\ 0 < c < d < \infty\}$ is a generating semiring for the Borel σ-field in $\mathbb{R} \times \mathbb{R}'_{+}$.

Q. E. D.

<u>Lemma 4.5.4</u> Suppose the condition Δ holds for $\{\xi_j\}$ and $\mathbb{N}_n$ converges in distribution to some point process $\mathbb{N}$. Then $\mathbb{N} \circ u_\tau \overset{d}{=} \mathbb{N}$, $\mathbb{N} \circ v_\sigma \overset{d}{=} \mathbb{N}$ for all $\tau \in \mathbb{R}$ and $\sigma > 0$.

Proof: First, we show $\mathbb{N} \circ u_\tau \overset{d}{=} \mathbb{N}$. By Theorem 2.2.2 (iii), it suffices to show for $U_1, U_2, \ldots, U_k \in P$,

$$(4.5.8) \quad (\mathbb{N}(U_1), \ldots, \mathbb{N}(U_k)) \overset{d}{=} (\mathbb{N}(u_\tau(U_1)), \ldots, \mathbb{N}(u_\tau(U_k))).$$

Note that for $U = [a,\ b) \times [c,\ d)$, $\mathbb{N}(U) \overset{d}{=} N^{(d)}([a,\ b)) - N^{(c)}([a,\ b))$ and $\mathbb{N}(u_\tau(U)) \overset{d}{=} N^{(d)}([a+\tau,\ b+\tau)) - N^{(c)}([a+\tau,\ b+\tau))$ by Theorem 4.4.2. Thus we only need to show that

$$(N^{(\tau_1)}, \ldots, N^{(\tau_k)}) \overset{d}{=} (N^{(\tau_1)}(\tau+\cdot), \ldots, N^{(\tau_k)}(\tau+\cdot))$$

for each $\tau_1, \tau_2, \ldots, \tau_k > 0$, which is readily seen from Theorem 4.2.3. By the same token, if we have v_σ instead of u_τ in (4.5.8), we would have to show

$$(4.5.9) \quad (N^{(\tau_1)}, \ldots, N^{(\tau_k)}) \overset{d}{=} (N^{(\tau_1/\sigma)}(\sigma\cdot), \ldots, N^{(\tau_k/\sigma)}(\sigma\cdot))$$

for each $\tau_1, \ldots, \tau_k > 0$. By Theorem 4.2.3, if $E\exp(-\Sigma_{j=1}^{k}\int_{\mathbb{R}} f_j dN^{(\tau_j)} = \exp(-\theta\tau_1\int(1 - L(f_1(t), \ldots, f_k(t)))dt)$, then

$$E\exp(-\Sigma_{j=1}^{k}\int_{\mathbb{R}} f_j(t)\, N^{(\tau_j/\sigma)}(\sigma dt)$$

$$= E\exp(-\Sigma_{j=1}^{k}\int_{\mathbb{R}} f_j(t/\sigma) N^{(\tau_j/\sigma)}(dt)$$

$$= \exp(-\frac{\theta\tau_1}{\sigma} \int (1 - L(f_1(t/\sigma), .., f_k(t/\sigma)))dt)$$

$$= \exp(-\theta\tau_1 \int (1- L(f_1(t), .., f_k(t)))dt)$$

$$= \mathrm{E}\exp -\Sigma_{j=1}^{k} \int_{\mathbb{R}} f_j dN^{(\tau_j)}$$

by a change of variable. The conclusion follows. Q. E. D.

Further, for each $\tau \in \mathbb{R}$ and $\sigma > 0$, let U_τ and V_σ be mappings from $N(\mathbb{R} \times \mathbb{R}_+')\setminus\{o\}$ to $N(\mathbb{R} \times \mathbb{R}_+')\setminus\{o\}$ defined by $U_\tau\colon \phi \to \phi \circ u_{-\tau}$ and $V_\sigma\colon \phi \to \phi \circ v_{1/\sigma}$ respectively; namely, if $\phi \in N(\mathbb{R} \times \mathbb{R}_+)\setminus\{o\}$ with a decomposition $\Sigma_i b_i \delta_{x_i}$, then $U_\tau(\phi) = \Sigma_i b_i \delta_{u_\tau(x_i)}$, $V_\sigma(\phi) = \Sigma_i b_i \delta_{v_\sigma(x_i)}$.

<u>Corollary 4.5.5</u> Write $\tilde{P}$ for the canonical measure of $\mathbb{N}$. Then $\tilde{P} \circ U_\tau = \tilde{P}$, $\tilde{P} \circ V_\sigma = \tilde{P}$ for all $\tau \in \mathbb{R}$ and $\sigma > 0$.

<u>Proof</u>: It is obvious that $\tilde{P} \circ U_\tau$ and $\tilde{P} \circ V_\sigma$ are the canonical measures of $\mathbb{N} \circ u_\tau$ and $\mathbb{N} \circ v_\sigma$ respectively. Since there is a one-to-one correspondence between the canonical measures and the distributions of infinitely divisible point processes, the result follows from the lemma. Q. E. D.

<u>Lemma 4.5.6</u> For $x \in \mathbb{R} \times \mathbb{R}_+'$, $\psi \in \overline{N}([1, \infty))$, we have

(4.5.10) $$U_\tau \circ g\ (x, \psi) = g\ (u_\tau x, \psi)$$

and

(4.5.11) $$V_\sigma \circ g\ (x, \psi) = g\ (v_\sigma x, \psi)$$

for every $\tau \in \mathbb{R}$ and $\sigma > 0$.

<u>Proof</u>: Let ψ have a decomposition $\Sigma_i a_i \varepsilon_{y_i}$, and $x = (s, t)$. Then $U_\tau \circ g(x, \psi) = U_\tau(\Sigma_i a_i \delta_{(s,\ ty_i)}) = \Sigma_i a_i \delta_{(s+\tau,\ ty_i)}$. On the other hand, $g(u_\tau x, \psi) = g((s+\tau, t), \psi) = \Sigma_i a_i \delta_{(s+\tau,\ ty_i)}$, proving (4.5.10). (4.5.11)

can be proved similarly. Q. E. D.

We now combine our lengthy and somewhat disconnected discussions to give the following result.

Theorem 4.5.7 Suppose that the condition Δ holds for $\{\xi_j\}$ and $\mathbb{N}_n$ converges in distribution to some point process $\mathbb{N}$. Then $\mathbb{N}$ is infinitely divisible with a canonical measure $\tilde{P}$ satisfying

$$\tilde{P} = \theta \cdot (Q \times m) \circ g^{-1}$$

where θ is the extremal index of $\{\xi_j\}$, m the Lebesque measure on $\mathbb{R} \times \mathbb{R}_+'$ and Q a probability measure on $(\overline{N}([1, \infty)), \overline{\mathcal{N}}([1, \infty)))$.

Proof: If $\theta = 0$, then the assertion is trivially true. Suppose $\theta \in (0, 1]$. For each $M \in \overline{\mathcal{N}}([1, \infty))$, define a measure ν_M on $\mathcal{B}(\mathbb{R} \times \mathbb{R}_+')$ by

$$(4.5.12) \qquad \nu_M(\cdot) = \tilde{P} \circ g(\cdot \times M).$$

for $B \in \mathcal{B}(\mathbb{R} \times \mathbb{R}_+')$,

$$\nu_M(u_\tau(B)) = \tilde{P} \circ g(u_\tau(B) \times M) = \tilde{P} \circ U_\tau \circ g(B \times M)$$

$$= \tilde{P} \circ g(B \times M) = \nu_M(B)$$

by Lemma 4.5.5 and Lemma 4.5.6. This shows that $\nu_M = \nu_M \circ u_\tau$, $\tau \in \mathbb{R}$. One could show similarly that $\nu_M = \nu_M \circ v_\sigma$, $\sigma > 0$. By Lemma 4.5.3, ν_M is thus a scalor multiple of Lebesque measure m; i.e,

$$(4.5.13) \qquad \nu_M(\cdot) = \theta \cdot Q(M) \cdot m(\cdot)$$

for some contant $Q(M) \in [0, \infty]$. It is clear that $Q(\phi) = 0$. Also if $\{M_i\}_{i=1}^{\infty}$ is a countable collection of disjoint sets in $\overline{\mathcal{N}}([1, \infty))$, then for

variables with the same distribution as ξ_1. If for each ε, $\nu > 0$,

$$\text{(5.3.10)} \qquad n\, P\{a_n \sup_{\lambda \geq n\nu} (c_\lambda Z_\lambda) > \varepsilon\} \xrightarrow{n\to\infty} 0,$$

$$n\, P\{a_n \sup_{\lambda \leq -n\nu} (c_\lambda Z_\lambda) > \varepsilon\} \xrightarrow{n\to\infty} 0,$$

then $\{\xi_t\}$ satisfies the condition $\Delta(u_{n,1}, \ldots, u_{n,k})$ with $u_{n,i} = x_i/a_n + b_n$ for arbitrary $x_1, \ldots, x_k$.

Proof: We only prove for the case $k = 1$. The extension to $k > 1$ is straightforward. Thus, let $u_n = x/a_n + b_n$, $G(x) > 0$. For a fixed $\nu \in (0, 1/2)$, let A, B be two events in the σ-fields

$$\mathcal{B}_1^k\, (u_n) = \sigma\{(\xi_t \leq u_n),\ t = 1, \ldots, k\},$$

$$\mathcal{B}_{k+2n\nu}^n\, (u_n) = \sigma\{(\xi_t \leq u_n),\ t = k+2n\nu, \ldots, n\}$$

respectively, where k is any number for which the above statements make sense. It is easy to see that A, B can be represented in the forms

$$A = \cup_{j=1}^{a}(\xi_t \in A_{t,j}^{(n)},\ t = 1, \ldots, k),$$

$$B = \cup_{j=1}^{b}(\xi_t \in B_{t,j}^{(n)},\ t = k+2n\nu, \ldots, n)$$

where $A_{t,j}^{(n)}$ and $B_{t,j}^{(n)}$ equal $(-\infty, u_n]$ or (u_n, ∞), $a, b < \infty$. Let

$$\xi_t' = \sup_{\lambda \leq n\nu-1} (c_\lambda Z_{\lambda+t}),$$

$$\xi_t'' = \sup_{\lambda \geq -n\nu+1} (c_\lambda Z_{\lambda+t}),$$

$$M_n' = \max_{1 \leq t \leq n} (|\xi_t - \xi_t'|),$$

$$M_n'' = \max_{1 \leq t \leq n} (|\xi_t - \xi_t''|).$$

For each $1 \le \lambda \le n$, $\beta_{n,\lambda} = c$ and thus $P\{\beta_{n,\lambda} Z_1 > u_n\} \sim c^{\alpha}\tau/n\Sigma_\lambda c_\lambda^\alpha$ by (5.3.3). This and the fact $\Lambda(y) \sim y^2/2$ as $y \to 0$ imply that

$$\Sigma_{\lambda=1}^{n} \Lambda(P\{\beta_{n,\lambda} Z_1 > u_n\}) \sim n/2 \ (c^\alpha\tau/n\Sigma_\lambda c_\lambda^\alpha)^2 \to 0 \text{ as } n \to \infty$$

This concludes the proof. Q. E. D.

Corollary 5.3.3 Let $\{a_n\}$ be a sequence of constants satisfying $a_n^\alpha\, L(a_n^{-1})\ \Sigma_\lambda\, c_\lambda^\alpha \sim 1/n$ as $n \to \infty$. Then

$$P\{a_n M_n \le x\} \overset{w}{\to} \exp(-c^\alpha/(\Sigma_\lambda c_\lambda^\alpha x^\alpha)),\ x > 0.$$

Proof: It is easily seen that

(5.3.8) $$P\{X_1 > a_n^{-1}\, \tau^{-1/\alpha}\} \sim \tau/n,$$

implying that $\{a_n^{-1}\, \tau^{-1/\alpha}\}$ is a "$u_n^{(\tau)}$-sequence". The conclusion follows from the proposition by letting $\tau = x^{-\alpha}$. Q. E. D.

The following result provides a convenient argument for verifying the condition $\Delta(u_n)$ for the max-moving average processes in general. The idea of the proof is based upon that of [34], Lemma 3.1.

Lemma 5.3.4 Let $\{\xi_t\}_{t\in I}$ be a max-moving average process; namely, $\xi_t = \sup_\lambda(c_{\lambda-t}\, Z_\lambda)$ for some sequence of constants $\{c_\lambda\}_{\lambda\in I}$ and i.i.d. random variables $\{Z_\lambda\}_{\lambda\in I}$. Here we impose conditions on neither $\{c_\lambda\}$ non the tail behavior of Z_1. Suppose there exists constants $a_n > 0$, b_n and non-degenerate distribution function G such that

(5.3.9) $$P\{a_n(\hat{M}_n - b_n) \le x\} \xrightarrow{n\to\infty} G(x)$$

for each x with $G(x) > 0$, where $\hat{M}_n$ is the maximum of n independent random

Proposition 5.3.2 $\{X_t\}$ has an extremal index $\theta = c^{\alpha}/\Sigma_{\lambda} c_{\lambda}^{\alpha}$, i.e.,

$\lim_{n\to\infty} P\{M_n \le u_n^{(\tau)}\} = \exp(-c^{\alpha}\tau/\Sigma_{\lambda} c_{\lambda}^{\alpha})$.

Proof: Again, we write u_n for $u_n^{(\tau)}$ and assume $c_0 = c$ for convenience. Since $X_t = \sup_{\lambda} c_{\lambda-t} Z_{\lambda}$, we have

$$
\begin{aligned}
P\{M_n \le u_n\} &= P\{\max_{1\le t\le n} \sup_{\lambda} c_{\lambda-t} Z_{\lambda} \le u_n\} \\
&= P\{\sup_{\lambda} \max_{1\le t\le n} c_{\lambda-t} Z_{\lambda} \le u_n\} \\
&= P\{\sup_{\lambda} \max_{\lambda-n\le t\le \lambda-1} c_t Z_{\lambda} \le u_n\} \\
&= \Pi_{\lambda} P\{\beta_{n,\lambda} Z_1 \le u_n\}.
\end{aligned}
$$

Hence for large n,

$$-\log P\{M_n \le u_n\} = \Sigma_{\lambda} P\{\beta_{n,\lambda} Z_1 > u_n\} + \Sigma_{\lambda}\Lambda(P\{\beta_{n,\lambda} Z_1 > u_n\})$$

where Λ is defined by (5.1.1). By Lemma 5.3.1.

$$\Sigma_{\lambda=1}^{n} P\{\beta_{n,\lambda} Z_1 > u_n\} \xrightarrow{n\to\infty} c^{\alpha}\tau/\Sigma_{\lambda} c_{\lambda}^{\alpha}$$

and

$$(5.3.7) \qquad \sum_{\substack{\lambda\in I \\ \lambda\notin\{1,2,..,n\}}} P\{\beta_{n,\lambda} Z_1 > u_n\} \xrightarrow{n\to\infty} 0.$$

Therefore it suffices to show that $\Sigma_{\lambda}\Lambda(P\{\beta_{n,\lambda} Z_1 > u_n\}) \xrightarrow{n\to\infty} 0$. By (5.3.7) and the definition of Λ, it is readily seen that

$$\sum_{\substack{\lambda\in I \\ \lambda\notin\{1,2,..,n\}}} \Lambda(P\{\beta_{n,\lambda} Z_1 > u_n\}) \to 0 \text{ as } n \to \infty.$$

First consider $\Sigma_{t=-\infty}^{-T+1} + \Sigma_{t=p_n+T}^{\infty} P\{\beta_{p_n,t} Z_1 > u_n\}$. For each $t \le -T+1$ (resp., $t \ge p_n + T$), $\beta_{p_n,t}$ equals some c_t, $t \le -T$ (resp., $t \ge T$); and for each $t \le -T$ (resp., $t \ge T$), c_t does not appear in the summation for more than p_n times. Thus

$$(5.3.5) \quad \Sigma_{t=-\infty}^{-T+1} + \Sigma_{t=p_n+T}^{\infty} P\{\beta_{p_n,t} Z_1 > u_n\}$$

$$\le p_n \Sigma_{|t|\ge T} P\{c_t Z_1 > u_n\}$$

$$\sim p_n L(u_n) \Sigma_{|t|\ge T} c_t^{\alpha} / u_n^{\alpha} \quad \text{by Lemma 5.2.1}$$

$$\sim p_n \tau \Sigma_{|t|\ge T} c_t^{\alpha} / n \Sigma_{\lambda} c_{\lambda}^{\alpha} \text{ by (5.3.1)}$$

$$\le p_n \tau \varepsilon / n \Sigma_{\lambda} c_{\lambda}^{\alpha}$$

by the choice of T. Also it follows simply from (5.3.3) that

$$(5.3.6) \quad \Sigma_{t=-T+2}^{0} + \Sigma_{t=p_n+1}^{p_n+T-1} P\{\beta_{p_n,t} Z_1 > u_n\}$$

$$\le 2(T-1) P\{cZ_1 > u_n\}$$

$$= o(p_n/n).$$

By (5.3.4), (5.3.5) and (5.3.6),

$$\limsup_{n\to\infty} n/p_n \sum_{\substack{t\in I \\ t\notin\{1,2,..,p_n\}}} P\{\beta_{p_n,t} Z_1 > u_n\} \le \tau\varepsilon/\Sigma_{\lambda} c_{\lambda}^{\alpha}.$$

The second assertion now follows since ε is arbitrary. Q. E. D.

and

$$\sum_{\substack{t\in I \\ t\notin\{1,2,..,p_n\}}} P\{\beta_{p_n,t}\, Z_1 > u_n^{(\tau)}\} = o(p_n/n).$$

Proof: To prove the first assertion, assume that the maximum c of c_λ is attained at $\lambda = k$. Thus for each fixed n,

$$\beta_{p_n,t} = c,\ t = k+1,\ k+2,\ ..,\ k+p_n.$$

By (5.3.1),

$$\text{(5.3.3)}\quad P\{cZ_1 > u_n\} = c^\alpha L(c^{-1}u_n)/u_n^\alpha \sim c^\alpha L(u_n)/u_n^\alpha \sim c^\alpha \tau/n\Sigma_\lambda c_\lambda^\alpha,$$

where we write u_n for $u_n^{(\tau)}$ for simplicity. Therefore

$$\Sigma_{t=k+1}^{k+p_n} P\{\beta_{p_n,t}\, Z_1 > u_n\} \sim p_n c^\alpha \tau/n\Sigma_\lambda c_\lambda^\alpha.$$

The first assertion now follows from

$$|\Sigma_{t=1}^{p_n} P\{\beta_{p_n,t}\, Z_1 > u_n\} - \Sigma_{t=k+1}^{k+p_n} P\{\beta_{p_n,t}\, Z_1 > u_n\}|$$

$$\leq 2kP\{cZ_1 > u_n\} = o(p_n P\{cZ_1 > u_n\}) = o(\Sigma_{t=k+1}^{k+p_n} P\{\beta_{p_n,t}\, Z_1 > u_n\})$$

since $p_n \to \infty$ as $n \to \infty$.

Next, for each $\varepsilon > 0$, there exists T such that $\Sigma_{|t|\geq T}\, c_t^\alpha < \varepsilon$. Let n be fixed, $\sum_{\substack{t\in I \\ t\notin\{1,2,..,p_n\}}} P\{\beta_{p_n,t}\, Z_1 > u_n\}$ can be written as

$$\text{(5.3.4)}\quad \Sigma_{t=-\infty}^{-T+1} + \Sigma_{t=-T+2}^{0} + \Sigma_{t=p_n+1}^{p_n+T-1} + \Sigma_{t=p_n+T}^{\infty} P\{\beta_{p_n,t}\, Z_1 > u_n\}.$$

$$P(X > x) \sim \Sigma_\lambda P(c_\lambda Z_\lambda > x) \sim x^{-\alpha} L(x) \Sigma_\lambda c_\lambda^\alpha$$

proving (5.2.2). Q. E. D.

For convenience, we assume henceforth, if not otherwise stated, that the moving average $\{X_t\}$ under consideration always satisfies one of (a), (b), (c).

5.3 The Extremal Properties of $\{X_t\}$

It is well known that X_1 belongs to the domain of attraction of the (max-) stable law $G(x) = \exp(-x^{-\alpha})$ (cf. [21], Theorem 1.6.2). It will be shown that M_n, when properly (linearly) normalized, has a limiting distribution which is of the same type as G. The mixing properties of high level crossings and the point processes considered in Chapter 3 and 4 will also be discussed.

We start with some notation, As before, write $\{u_n^{(\tau)}\}$ for a sequence satisfying $P(X_1 > u_n^{(\tau)}) \sim \tau/n$, or, by (5.2.2),

$$(5.3.1) \qquad (L(u_n^{(\tau)})/(u_n^{(\tau)})^\alpha)\Sigma_\lambda c_\lambda^\alpha \sim \tau$$

Let $c = \max_\lambda c_\lambda$. For each $t \in I$ and $\ell = 1, 2, 3, ..,$ write

$$(5.3.2) \qquad \beta_{\ell,t} = \max_{t-\ell\le\lambda\le t-1} (c_\lambda).$$

The following result is useful.

<u>Lemma 5.3.1</u> Let $p_1, p_2, \ldots$ be positive integers, $p_n \to \infty$ as $n \to \infty$. Then

$$\Sigma_{t=1}^{p_n} P\{\beta_{p_n,t} Z_1 > u_n^{(\tau)}\} \sim \frac{p_n \tau c^\alpha}{n\Sigma_\lambda c_\lambda^\alpha}$$

$$L(\rho x)/L(x) \le 2 \exp(\varepsilon \int_{\rho}^{1} du/u) = 2\rho^{-\varepsilon} \le 2\rho_0^{-\varepsilon}.$$

We can thus take K to be $2\rho_0^{-2\varepsilon}$ and this concludes the proof. Q. E. D.

<u>Proposition 5.2.2</u> Let $\{X_t\}$ be as defined in (5.2.1) where the c_λ are non-negative and $P(Z_1 > z) = z^{-\alpha}L(z)$, L being slowly varying at ∞ and α positive. Then X_1 is almost surely finite and, in fact,

$$(5.2.2) \qquad P(X_1 > x) \sim x^{-\alpha} L(x) \Sigma_\lambda c_\lambda^\alpha$$

as $x \to \infty$ provided that any one of the following (a) (b) or (c) holds:

(a) $\Sigma_\lambda c_\lambda^{\alpha-\varepsilon} < \infty$ for some $0 < \varepsilon < \alpha$;

(b) $\Sigma_\lambda c_\lambda^{\alpha} < \infty$ and L is eventually non-increasing;

(c) $\Sigma_\lambda c_\lambda^{\alpha} < \infty$ and $L(x)$ converges to some positive constant as x tends to ∞.

<u>Proof</u>: We only prove the assertion under (a). The proofs under (b) and (c) are more straightforward. The assumption implies that $c \overset{\text{def}}{=} \max(c_\lambda) < \infty$. Since $c_\lambda^{-1} \ge c > 0$ for each λ, Lemma 5.2.1 implies that there exist x_0 and K such that

$$\Sigma_\lambda P(c_\lambda Z_\lambda > x)/L(x) = x^{-\alpha} \Sigma_\lambda c_\lambda^\alpha L(c_\lambda^{-1} x)/L(x)$$

$$\le K x^{-\alpha} \Sigma_\lambda c_\lambda^{\alpha-\varepsilon}$$

for each x x_0, where we interpret c $L(c^{-1} x)$ as zero if c $= 0$. Hence $\Sigma_\lambda P(c_\lambda Z_\lambda > x) \sim x^{-\alpha} L(x) \Sigma_\lambda c_\lambda^\alpha$ by dominated convergence and the fact $L(tx)/L(x) \xrightarrow{x\to\infty} 1$ for each fixed $t > 0$. Theorem 5.1.1 now implies that $X_1 < \infty$ a.s., and, consequently, the distribution function of X_1 does not have a jump at its right end point since the right end point obviously equals infinity. Thus Lemma 5.1.2,

and $\varepsilon > 0$, there exist x_0 and K such that

$$L(\rho x)/L(x) < K\rho^{\varepsilon} \text{ for all } \rho \geq \rho_0,\ x \geq x_0.$$

Proof: It is known (cf. [11]) that L can be represented as

$$L(t) = a(t) \exp(\int_1^t \varepsilon(u)/u \, du)$$

where a(t) is a positive, bounded and measurable function that converges to some positive constant as $t \to \infty$, and $\varepsilon(t)$ is a continuous function that tends to zero as $t \to \infty$. We can assume without loss of generality that $0 < \rho_0 < 1$. It is easily seen that there exists an x_0 such that for each $\rho \geq \rho_0$ and $x \geq x_0$ we have

$$a(\rho x)/a(x) < 2$$

and

$$|\varepsilon(\rho x)| < \varepsilon.$$

Thus for $\rho \geq \rho_0$ and $x \geq x_0$,

$$\begin{aligned} L(\rho x)/L(x) &= a(\rho x)/a(x) \exp(\int_x^{\rho x} \varepsilon(u)/u \, du) \\ &= a(\rho x)/a(x) \exp(\int_1^{\rho} \varepsilon(ux)/u \, du) \\ &\leq 2\exp(\varepsilon|\int_1^{\rho} du/u|) \end{aligned}$$

where we interpret $\int_a^b f(u)du$ as $-\int_b^a f(u)du$ if $b < a$. Consequently for each $\rho \geq 1$ and $x \geq x_0$,

$$L(\rho x)/L(x) \leq 2 \exp(\varepsilon \int_1^{\rho} du/u) = 2\rho^{\varepsilon},$$

and for $\rho_0 \leq \rho < 1$ and $x \geq x_0$,

5.2 Framework

Let $\{c_\lambda\}_{-\infty}^{\infty}$ be a sequence of constants which are not all zero. Define a stationary sequence of random variables $\{X_t\}_{-\infty}^{\infty}$ by

(5.2.1) $$X_t = \sup_\lambda c_{\lambda-t} Z_\lambda$$

where $\{Z_\lambda\}_{-\infty}^{\infty}$, the noise sequence, consists of independent and identically distributed random variables. For convenience, call $\{X_t\}$ a max-moving average process. It is interesting to note the parallels between $\{X_t\}$ and the usual moving average. We shall see, with certain tail assumptions, the extremal behaviors of the two are strikingly similar.

A function L is said to be slowly varying at ∞ if it is positive and measurable on $(0, \infty)$ with $\lim_{x\to\infty} L(tx)/L(x) = 1$ for each $t > 0$. A function R is said to be regularly varying at ∞ with index α if $R(x) = x^\alpha L(x)$, $x > 0$, where L is some function slowly varying at ∞. Naturally,

$$\lim_{x\to\infty} R(tx)/R(x) = t^\alpha$$

Some helpful references concerning slowly and regularly varying functions are [11] and [12].

Throughout this and the later sections, our study of the max-moving average process will be confined to the special case where the coefficients c_λ's are non-negative and $P(Z_1 > z)$ is regularly varying at ∞ with index $-\alpha$, $\alpha > 0$. For $\{X_t\}$ thus defined, two immediate questions are:

(a) Is $X_1 < \infty$ a.s. ?

(b) Is $P(X_1 > x)$ regularly varying at ∞?

The following technical lemma provides an anwser.

<u>Lemma 5.2.1</u> Let L be a slowly varying function. For any fixed $\rho_0 > 0$

Conversely, suppose $X < \infty$ a.s.. Then there exists an x such that $P(X \leq x) > 0$, which implies that $-\log P(X \leq x) < \infty$. By independence and (5.1.1),

$$-\log P(X \leq x) = -\log \Pi_i P(Y_i \leq x) = -\Sigma_i \log P(Y_i \leq x)$$

$$= \Sigma_i P(Y_i > x) + \Sigma_i \Lambda(P(Y_i > x)).$$

But this implies that $\Sigma_i P(Y_i > x) < \infty$ since $-\log P(X \leq x) < \infty$ and $\Sigma_i \Lambda(P(Y_i > x)) \geq 0$. Q. E. D.

<u>Corollary 5.1.2</u> Write $x_0 = \sup\{u: P(X \leq u) < 1\}$. If $P(X < x_0) = 1$, then $P(X > u) \sim \Sigma_i P(Y_i > u)$ as $u \to x_0$.

<u>Proof</u>: Since $P(X < x_0) = 1$, we can find $x < x_0$ for which $P(X \leq x) > 0$. Hence for $x \leq u \leq x_0$,

$$-\log P(X \leq u) = \Sigma_i P(Y_i > u) + \Sigma_i \Lambda(P(Y_i > u)).$$

The assumption $P(X < x_0) = 1$ implies $P(X \leq u) \to 1$ as $u \to x_0$. As a consequence,

$$-\log P(X \leq u) \sim P(X > u)$$

and

$$\Sigma_i P(Y_i > u) \to 0 \text{ as } u \to x_0.$$

The result now follows from the fact $\Sigma\Lambda(P(Y_i > u)) = o(\Sigma P(Y_i > u))$.

Q. E. D.

CHAPTER V

EXTREME VALUE THEORY FOR THE SUPREMUM OF WEIGHTED RANDOM VARIABLES WITH REGULARLY VARYING TAIL PROBABILITIES

5.1 The Supremum of a Sequence of Independent Random Variables

To demonstrate the notions mentioned previously, we now stady a class of processes which is interesting in its own right.

Throughout this chapter, a random variable bears the meaning of an extended real-valued random variable, i.e., it is a measurable mapping from some probability space to the extended real line $\mathbb{R}^* = [-\infty, \infty]$. The following result is basic.

Theorem 5.1.1 (cf. [8]) Let Y_1, Y_2, ... be a.s. finite and mutually independent random variables defined on some probability space. Write $X = \sup Y_i$. Then $X < \infty$ a.s. if and only if $\Sigma_i P(Y_i > x) < \infty$ for some $x < \infty$. This shows, in particular, that $X = \infty$ a.s. or $x < \infty$ a.s..

Proof: write

$$(5.1.1) \qquad \Lambda(y) = y + \log(1 - y), \; 0 \leq y < 1.$$

It follows by Taylor's Theorem that $\Lambda(y) \geq 0$, $\Lambda(y) \sim y^2/2$ as $y \to \infty$. Suppose first that $\Sigma_i P(Y_i > x) < \infty$ for some $x < \infty$. Then

$$\lim_{u\to\infty} P(X > u) \leq \lim_{u\to\infty} \Sigma_i \, P(Y_i > u) = 0$$

by Boole's inequality and dominated convergence. Thus $X < \infty$ a.s..

On the other hand, Faton's Lemma implies that

$$E\,\mathbb{N}((0, 1) \times (0, \tau)) \leq \liminf E\,\mathbb{N}_n((0, 1) \times (0, \tau)) = \tau.$$

If follows that $E\,\mathbb{N}((0, 1) \times (0, \tau)) = \tau$ and $k_1 = 1$ a.s. Q. E. D.

It is important to observe the differences between Mori's and our result. Mori assumes that $\{\xi_j\}$ is strongly mixing and considers the point process $\Sigma\delta_{(j/n,\ a_n^{-1}(\xi_j - b_n))}$. We assumed the condition Δ and considered the point process $\Sigma\delta_{(j/n,\ u_n^{-1}(\xi_j))}$. It is quite obvious that neither result contains the other. However, the two are similar when the extremal index θ exists in $(0, 1]$ and ξ_1 belongs to the domain of attraction of some max-stable law. We feel that it is possible to have a unified approach using normalizations that are more general than the ones in both results. To be more specific, we propose to study the point process $\Sigma\delta_{(j/n,\ u_n^{-1}(\xi_j))}$ where $\{f_n\}$ is a sequence of measurable functions such that $P(M_n \leq u_n^{(\tau)}) \to e^{-\tau}$, $\tau > 0$, as $n \to \infty$. Here we neither assume that u_n is linear nor require that $1 - F(u_n^{(\tau)}) \sim \tau/n$. This is certainly the direction of future endeavor.

The intimate relationship between the limit point processes and Poisson Processes is simply seen as follows.

Corollary 4.5.8 Let $\{(S_i, T_i), i = 1, 2, ..\}$ be the points of a homogeneous Poisson Process ζ on $\mathbb{R} \times \mathbb{R}_+'$ with mean θ. On the same probability space, let $\eta_1, \eta_2, \ldots$ be a sequence of identically distributed random elements in $(\overline{N}([1, \infty)), \overline{N}([1, \infty)))$ with common distribution Q, and let $\{1 = Y_{i1} \leq Y_{i2} \leq \ldots \leq Y_{ik_i}\}$ be the points of η_i, $i = 1, 2, ..,$ where $k_1, k_2, \ldots$ are r.v.'s in $\{1, 2, .., \infty\}$. Assume that the η_i are independent of ζ and are themselves independent. Then

$$\mathbb{N} \stackrel{d}{=} \Sigma_{i=1}^{\infty} \Sigma_{j=1}^{k_i} \delta_{(S_i, T_i Y_{ij})}. \tag{4.5.14}$$

Proof: It suffices to compare the Laplace Transforms of the two point processes in (4.5.14) Q. E. D.

An extremely pleasant situation is when η_1 is degenerate; i.e. the atoms of η_1 are fixed with probability one. The following is well known (cf. [21], Theorem 5.7.1).

Corollary 4.5.9 Suppose θ, the extremal index, equals one. Then $\mathbb{N}$ is Poisson with mean 1.

Proof: Using the notation of Corollary 5.6, it can be seen that

$$E\,\mathbb{N}((0, 1) \times (0, \tau))$$

$$= E[\zeta((0, 1) \times (0, \tau))] \cdot E[\Sigma_{j=1}^{k_1} \int_{u=0}^{\tau} P(Y_{i_j} u < \tau) d(u/\tau)]$$

$$= \tau E[\Sigma_{j=1}^{k_1} \int_0^1 P(Y_{1j} < 1/x) dx] \geq \tau$$

each $B \in \mathcal{B}(\mathbb{R} \times \mathbb{R}'_+)$ for which $m(B) > 0$, we have

$$Q(\cup M_i) = \tilde{P} \circ g(B \times \cup M_i)/\theta m(B) = \Sigma \tilde{P} \circ g(B \times M_i)/\theta m(B) = \Sigma Q(M_i)$$

since g is one-to-one. This shows that Q is a measure on $\overline{N}([1, \infty))$. Moreover, note that g maps the set $([0, 1) \times (0, \tau)) \times \overline{N}([1, \infty))$ to $\{\phi \in N(\mathbb{R} \times \mathbb{R}'_+)\setminus\{o\}: \phi([0, 1) \times (0, \tau)) > 0\} \cap \Lambda$, and

$$\tilde{P}\{\phi \in N(\mathbb{R} \times \mathbb{R}'_+)\setminus\{o\}: \phi([0, 1) \times (0, \tau)) > 0\}$$

$$= -\log P\{\mathbb{N}([0, 1) \times (0, \tau)) = 0\}$$

$$= -\log P\{N^{(\tau)}([0, 1)) = 0\}$$

$$= \theta\tau$$

by Lemma 2.2.3 and Theorem 4.3.4. Consequently,

$$Q(\overline{N}([1, \infty))) = \tilde{P} \circ g(([0, 1) \times (0, \tau)) \times \overline{N}([1, \infty)))/\theta m([0, 1) \times (0, \tau))$$

$$= 1,$$

showing that Q is a probability measure. We have thus shown, by (4.5.12) and (4.5.13), that

$$\tilde{P} \circ g \circ = \theta \cdot (Q \times m).$$

Finally for each set E in the Borel σ-field of $N(\mathbb{R} \times \mathbb{R}'_+)\setminus\{o\}$,

$$\tilde{P} \circ g \circ (g^{-1}E) = \theta \cdot (Q \times m)(g^{-1}E).$$

The left hand side is simply $\tilde{P}(E \cap \Lambda)$, which equals $\tilde{P}(E)$ since $\tilde{P}$ is concentrated on Λ. Thus

$$\tilde{P} = \theta \cdot (Q \times m)\ g^{-1}. \qquad \text{Q. E. D.}$$

Clearly, ξ_i' and ξ_j'' are independent if $j - i \geq 2n\nu$. Therefore for $\varepsilon > 0$,

$$P(A \cap B) \leq P\{\cup_{j=1}^{a}(\xi_t^1 \in \tilde{A}_{t,j}^{(n)},\ t = 1, \ldots, k)\} \cdot P\{\cup_{j=1}^{b}(\xi_t'' \in \tilde{B}_{t,j}^{(n)},$$

$$t = k+2n\nu, \ldots, n\} + P\{M_n' > \varepsilon\} + P\{M_n'' > \varepsilon\}$$

where $\tilde{A}_{t,j}^{(n)}$ (or $\tilde{B}_{t,j}^{(n)}$) $= (-\infty, u_n+\varepsilon]$ or $(u_n-\varepsilon, \infty)$ depending upon $A_{t,j}^{(n)}$ (or $B_{t,j}^{(n)}$) $= (-\infty, u_n]$ or (u_n, ∞). Thus

$$P(A \cap B)$$

$$\leq P\{\cup_{j=1}^{a}\ (\xi_t \in \tilde{\tilde{A}}_{t,j}^{(n)},\ t=1, \ldots, k)\} \cdot P\{\cup_{j=1}^{b}\ (\xi_t \in \tilde{\tilde{B}}_{t,j}^{(n)},\ t=k+2n\ , \ldots, n)\}$$

$$+ 2P\{M_n' > \varepsilon\} + 2P\{M_n'' > \varepsilon\}$$

where $\tilde{\tilde{A}}_{t,j}^{(n)}$ (or $\tilde{\tilde{B}}_{t,j}^{(n)}$) $= (-\infty, u_n+2\varepsilon]$ or $(u_n-2\varepsilon, \infty)$ depending upon $A_{t,j}^{(n)}$ (or $B_{t,j}^{(n)}$) $= (-\infty, u_n]$ or (u_n, ∞). We therefore have

$$P(A \cap B) \leq P(A)P(B) + \Sigma_{t=1}^{n}\ P\{u_n-2\varepsilon < \xi_t \leq u_n+2\varepsilon\} +$$

$$2P\{M_n' > \varepsilon\} + 2P\{M_n'' > \varepsilon\}.$$

A corresponding lower limit can be obtained similarly. As a consequence,

$$|P(A \cap B) - P(A)P(B)| \leq nP\{u_n-2\varepsilon < \xi_0 \leq u_n+2\varepsilon\}$$

$$+ 2nP\{|\xi_0 - \xi_0'| > \varepsilon\} + 2nP\{|\xi_0 - \xi_0''| > \varepsilon\}.$$

The quantity on the right-hand side is independent of the choices of k, A and B. Replace ε by ε/a_n and writing $u_n + 2\varepsilon/a_n = x+2\varepsilon/a_n + b_n$, etc., we have that

$$\Delta_{\nu,n} \triangleq \sup(|P(A \cap B) - P(A)P(B)|:\ A \in \mathcal{B}_1^k(u_n),\ B \in \mathcal{B}_{k+2n\nu}^{n}(u_n),\ k \leq$$

$$(1-2\nu)n) \le nP\{(x-2\varepsilon)/a_n + b_n < \xi_0 \le (x+2\varepsilon)/a_n + b_n\}$$

$$+ 2nP\{a_n \sup_{\lambda \ge n\nu} (c_\lambda Z_\lambda) > \varepsilon\} + 2nP\{a_n \sup_{\lambda \le -n\nu} (c_\lambda Z_\lambda) > \varepsilon\}.$$

The first term converges to log $G(x+2\varepsilon)$ - log $G(x-2\varepsilon)$ according to [21], Theorem 1.5.1. The second and third term converges to zero by assumption (5.3.10). Thus, by letting $\varepsilon \to 0$, we have

$$\Delta_{\nu,n} \to 0 \text{ as } n \to \infty$$

since G is an extreme value distribution and hence continuous. This shows that $\Delta(u_n)$ holds since $\nu \in (0, 1/2)$ is arbitrary by a variant of [21] Lemma 3.2.1. Q. E. D.

Corollary 5.3.5 Let $\{X_t\}$ be a max-moving average process as described in Proposition 5.2.2. Then $\{X_t\}$ satisfies the condition $\Delta(u_n^{(\tau_1)}, \ldots, u_n^{(\tau_k)})$ for each k and each choice of positive $\tau_1, \ldots, \tau_k$; i.e. the condition Δ holds for $\{X_t\}$.

Proof: By corollary 5.3.2, $P\{a_n M_n \le x\}$ converges weakly to the distribution function $\exp(-c^\alpha/(\Sigma_\lambda c_\lambda^\alpha x^\alpha))$, $x > 0$, where a_n satisfies

$$a_n^\alpha L(a_n^{-1})\Sigma_\lambda c_\lambda^\alpha \sim 1/n. \tag{5.3.11}$$

Thus if for each ε, $\nu > 0$,

$$nP\{a_n \sup_{\lambda \ge n\nu} (c_\lambda Z_\lambda) > \varepsilon\} \xrightarrow{n\to\infty} 0, \tag{5.3.12}$$

and

$$nP\{a_n \sup_{\lambda \le -n\nu} (c_\lambda Z_\lambda) > \varepsilon\} \xrightarrow{n\to\infty} 0, \tag{5.3.13}$$

then $\{X_t\}$ satisfies the condition $\Delta(u_{n,1}, \ldots, u_{n,k})$ with $u_{n,i} = x_i/a_n$ for

arbitrary $x_1, \ldots, x_n$, which concludes the proof since $P\{X_1 > a_n^{-1}\tau^{-1/2}\} \sim \tau/n$, $\tau > 0$ (cf. (5.2.11)). Hence it suffices to show (5.3.12) and (5.3.13). Here we only verify (5.3.12) for the case where there exists $0 < \delta < \alpha$ for which $\Sigma_\lambda c_\lambda^{\alpha-\delta} < \infty$, the rest being similar. By Lemma 5.2.1, there exist n_0 and k such that for $n \geq n_0$,

$$(5.3.14) \quad \Sigma_{\lambda \geq n\nu} P\{a_n c_\lambda Z_1 > \varepsilon\} = \Sigma_{\lambda \geq n\nu} a_n^\alpha c_\lambda^\alpha \varepsilon^{-\alpha} L(a_n^{-1} c_\lambda^{-1} \varepsilon)$$

$$\leq K a_n^\alpha \varepsilon^{\delta-\alpha} L(a_n^{-1}) \Sigma_{\lambda \geq n\nu} c_\lambda^{\alpha-\delta}$$

since $a_n^{-1} \to \infty$. (5.3.14) and (5.3.11) imply that for large n

$$n \Sigma_{\lambda \geq n\nu} P\{a_n c_\lambda Z_1 > \varepsilon\} \leq K \varepsilon^{\delta-\alpha} \Sigma_{\lambda \geq n\nu} c_\lambda^{\alpha-\delta} / \Sigma_\lambda c_\lambda^\alpha,$$

which tends to zero as n tends to ∞ since $\Sigma_\lambda c_\lambda^{\alpha-\delta} < \infty$. The assertion (5.3.12) now follows by Boole's inequality. Q. E. D.

We now examine the "local" behavior of $\{X_t\}$. The most important idea involved in the following proof is roughly that a cluster of exceedances of a high level by $\{\xi_j\}$ is the consequence of a single large "Z" from the noise sequence-a property shared by the usual moving average (cf. [34]). Since the c_λ are not assumed to be all different, it is convenient to introduce the following. Define $\{\lambda_j\}_{j=0}^\infty$ inductively by

$$(5.3.15) \quad \lambda_0 = \infty, \ \lambda_1 = \max\{k: c_k = \max_{\lambda \in I} (c_\lambda)\},$$

$$\lambda_j = \max\{k: c_k = \max_{\lambda \in I \setminus \{\lambda_1, \ldots, \lambda_{j-1}\}} (c_\lambda)\}, \ j = 2, 3, 4, \ldots .$$

Obviously, $c_{\lambda_1} \geq c_{\lambda_2} \geq c_{\lambda_3} \geq \ldots$.

Lemma 5.3.6 Suppose $\tau_1 > \tau_2 > .. > \tau_J > 0$ are constants, and $\{r_n\}$ is a sequence such that $r_n = o(n)$ and $r_n \to \infty$ as $n \to \infty$. For large n, let $\mathcal{J}_{n,i}$, $i \geq 1$, be the interval

$$[1 + \max_{1\leq j\leq i+1} (\lambda_j),\ r_n + \min_{1\leq j\leq i+1} (\lambda_j)]$$

and $E_{n,i}$ the event

$$\cup_{k\in\mathcal{J}_{n,i}} (\beta_{r_n,k} Z_k > u_n^{(\tau_1)},\ \beta_{r_n,i} Z_i \leq u_n^{(\tau_1)} \text{ for all } i \neq k).$$

Then for each $(i_1, \ldots, i_J) \in I_J$,

$$P\{\Sigma_{m=1}^{r_n} X_{n,m}^{(\tau_j)} = i_j,\ j = 1, \ldots, J,\ E_{n,i_1}\}$$

$$\sim n \cdot (r_n \Sigma_\lambda c_\lambda^\alpha)^{-1} \cdot [\min_{1\leq j\leq J} (\tau_j c_{\lambda_{i_j}}^\alpha) - \max_{1\leq j\leq J} (\tau_j c_{\lambda_{i_j+1}}^\alpha)]^+$$

where $x^+ = \max(x, 0)$.

Proof: Assume that $\min_{1\leq j\leq J} (\tau_j c_{\lambda_{i_j}}^\alpha) > \max_{1\leq j\leq J} (\tau_j c_{\lambda_{i_j+1}}^\alpha)$, and that $c_{\lambda_{i_1+1}} > 0$, which implies $c_{\lambda_j} > 0$ for all $1 \leq j \leq i_1$; the modification needed is obvious if otherwise. If $k \in \mathcal{J}_{n,i_1}$, then $k-1 \geq \max_{1\leq j\leq i_1+1} (\lambda_j)$ and $k-r_n \leq \min_{1\leq j\leq i_1+1} (\lambda_j)$. This implies that $\{c_{\lambda_1}, c_{\lambda_2}, \ldots, c_{\lambda_{i_1+1}}\} \subset \{c_{k-r_n}, c_{k-r_n+1}, \ldots, c_{k-1}\}$ and, consequently, for each $1 \leq r \leq i_1+1$, the r-th largest among $c_{k-r_n}, \ldots, c_{k-1}$ equals c_{λ_r}. Now

$$P\{\Sigma_{m=1}^{r_n} X_{n,m}^{(\tau_j)} = i_j,\ j = 1, \ldots, J,\ E_{n,i_1}\}$$

$$= \Sigma_{k\in\mathcal{J}_{n,i_1}} P\{\Sigma_{m=1}^{r_n} X_{n,m}^{(\tau_j)} = i_j,\ j = 1, \ldots, J,\ \beta_{r_n,k} Z_k > u_n^{(\tau_1)},$$

$$\beta_{r_n,i} Z_i \le u_n^{(\tau_1)} \text{ for all } i \ne k\}$$

$$= \Sigma_{k\in \mathscr{I}_{n,i_1}} P\{c_{\lambda_{i_j+1}} Z_k \le u_n^{(\tau_j)} < c_{\lambda_{i_j}} Z_k,\ j = 1, .., J,$$

$$\beta_{r_n,i} Z_i \le u_n^{(\tau_1)} \text{ for all } i \ne k\}$$

$$= \Sigma_{k\in \mathscr{I}_{n,i_1}} P\{c_{\lambda_{i_j+1}} Z_k \le u_n^{(\tau_j)} < c_{\lambda_{i_j}} Z_k,\ j = 1, .., J\} P\{\beta_{r_n,i} Z_i$$

$$\le u_n^{(\tau_1)} \text{ for all } i \ne k\}$$

By stationarity and the fact (cf. Lemma 5.3.1)

$$\lim_{n\to\infty} P\{\beta_{r_n,i} Z_i \le u_n^{(\tau_1)} \text{ for all } i\} = 1,$$

we have

$$\text{(5.3.16)} \qquad P\{\Sigma_{m=1}^{r_n} \chi_{n,m}^{(\tau_j)} = i_j,\ j = 1, .., J, E_{n,i_1}\}$$

$$\sim r_n P\{c_{\lambda_{i_j+1}} Z_1 \le u_n^{(\tau_j)} < c_{\lambda_{i_j}} Z_1,\ j = 1, .., J\}$$

$$= r_n P\{\max_{1\le j\le J} u_n^{(\tau_j)}/c_{\lambda_{i_j}} < Z_1 \le \min_{1\le j\le J} u_n^{(\tau_j)}/c_{\lambda_{i_j+1}}\}$$

since $L(u_n^{(\tau)})/(u_n^{(\tau)})^\alpha \sim \tau/n\Sigma_\lambda c_\lambda^\alpha$ by (5.3.1), it follows that

$$P\{\max_{1\le j\le J} (u_n^{(\tau_j)}/c_{\lambda_{i_j}}) < Z_1 \le \min_{1\le j\le J} (u_n^{(\tau_j)}/c_{\lambda_{i_j+1}})\}$$

$$= \min_{1\le j\le J} P\{Z_1 > u_n^{(\tau_j)}/c_{\lambda_{i_j}}\} - \max_{1\le j\le J} P\{Z_1 > u_n^{(\tau_j)}/c_{\lambda_{i_j}+1}\}$$

$$= \min_{1\le j\le J} c_{\lambda_{i_j}}^\alpha L(u_n^{(\tau_j)}/c_{\lambda_{i_j}})/(u_n^{(\tau_j)})^\alpha - \max_{1\le j\le J} c_{\lambda_{i_j+1}}^\alpha L(u_n^{(\tau_j)}/c_{\lambda_{i_j+1}})/(u_n^{(\tau_j)})^\alpha$$

$$\sim (\min_{1\le j\le J} \tau_j c^\alpha_{\lambda_{i_j}} - \max_{1\le j\le J} \tau_j c^\alpha_{\lambda_{i_j+1}})/n\Sigma_\lambda c^\alpha_\lambda.$$

This together with (5.3.15) conclude the proof. Q. E. D.

Proposition 5.3.7 Let $\tau_1 > \tau_2 > .. > \tau_J > 0$ be constants and $\{r_n\}$ be $\Delta(u_n^{(\tau_1)}, \ldots, u_n^{(\tau_J)})$-separating. Then for each $(i_1, \ldots, i_J) \in I_J$,

(5.3.17)
$$\lim_{n\to\infty} P\{\Sigma_{m=1}^{r_n} \chi_{n,m}^{(\tau_j)} = i_j,\ j = 1, \ldots, J \mid \Sigma_{m=1}^{r_n} \chi_{n,m}^{(\tau_1)} > 0\}$$
$$= [\min_{1\le j\le J} (\tau_j c^\alpha_{\lambda_{i_j}} /\tau_1 c^\alpha) - \max_{1\le j\le J} (\tau_j c^\alpha_{\lambda_{i_j+1}} /\tau_1 c^\alpha)]^+$$

where $x^+ = \max(x, 0)$.

Proof: First note that

(5.3.18)
$$0 \le P\{\Sigma_{m=1}^{r_n} \chi_{n,m}^{(\tau_j)} = i_j,\ j = 1, \ldots, J \mid \Sigma_{m=1}^{r_n} \chi_{n,m}^{(\tau_1)} > 0\}$$
$$- P\{\Sigma_{m=1}^{r_n} \chi_{n,m}^{(\tau_j)} = i_j,\ j = 1, \ldots, J,\ E_{n,i_1} \mid \Sigma_{m=1}^{r_n} \chi_{n,m}^{(\tau_1)} > 0\}$$
$$= P\{\Sigma_{m=1}^{r_n} \chi_{n,m}^{(\tau_j)} = i_j,\ j = 1, \ldots, J,\ E^c_{n,i_1} \mid \Sigma_{m=1}^{r_n} \chi_{n,m}^{(\tau_1)} > 0\}$$
$$\le (P\{\beta_{r_n,i} Z_i > u_n^{(\tau_1)} \text{ for some } i\in\{1,..,r_n\}\} + P\{\beta_{r_n,i} Z_i$$
$$> u_n^{(\tau_1)} \text{ for more than one } i \text{ in } \{1,..,r_n\}\})/P\{\Sigma_{m=1}^{r_n} \chi_{n,m}^{(\tau_1)} > 0\}.$$

By Lemma 5.3.6, the proposition would have been proved if we could show the right hand side of (5.3.18) tends to zero since (cf. Lemma 5.3.1)

(5.3.19)
$$P\{\Sigma_{m=1}^{r_n} \chi_{n,m}^{(\tau_1)} > 0\} \sim c^\alpha.\tau_1.r_n/(n\Sigma_\lambda c^\alpha_\lambda).$$

By Boole's inequality and Lemma 5.3.1,

$$(5.3.20) \qquad P\{\beta_{r_n,i}\, Z_i > u_n^{(\tau_1)} \text{ for some } i \in \{1, .., r_n\}\}$$

$$\leq \sum_{\substack{i \in I \\ i \notin \{1,..,r_n\}}} P\{\beta_{r_n,i}\, Z_1 > u_n^{(\tau_1)}\}$$

$$= o(r_n/n).$$

Also we have

$$(5.3.21) \quad P\{\beta_{r_n,i}\, Z_i > u_n^{(\tau_1)} \text{ for more than one } i \text{ in } \{1, ..., r_n\}\}$$

$$\leq \binom{r_n}{2} [P\{cZ_1 > u_n^{(\tau_1)}\}]^2$$

$$\leq (r_n P\{Z_1 > u_n^{(\tau)}/c\})^2/2$$

$$\sim (c^\alpha \tau\, r_n/(n\Sigma_\lambda c_\lambda^\alpha))^2/2$$

$$= o(r_n/n).$$

Combining (5.3.18), (5.3.19) and (5.3.20), the conclusion follows.

Q. E. D.

Combining proposition 5.3.7, Corollary 5.3.5, Proposition 5.3.2, the following is immediate by Theorem 4.2.4.

Proposition 5.3.8 Let $\{X_t\}$ be as described in Proposition 5.2.2. Using the notation of Chapter IV, for each choice of constants $\infty > \tau_1$ $\tau_2 > .. > \tau_J > 0$, the point process $(N_n^{(\tau_1)}, ..., N_n^{(\tau_k)})$ converges in distribution to some point process $(N^{(\tau_1)}, ..., N^{(\tau_k)})$ with Laplace Transform

$$\exp\left(- \frac{c^\alpha\ \tau_1}{\Sigma_\lambda\ c_\lambda^\alpha} \int_{\mathbb{R}} (1 - L(f_1(t), ..., f_J(t)))dt\right)$$

where

$$L(s_1, \ldots, s_J) = \Sigma_{(i_1, \ldots, i_J)} \, {}_J\Big(\min_{1 \le j \le J} (\tau_j c^{\alpha}_{\lambda_{i_j}} / \tau_1 c^{\alpha}) -$$

$$\max_{1 \le j \le J} (\tau_j c^{\alpha}_{\lambda_{i_j+1}} / \tau_1 c^{\alpha})\Big)^{+} \times \exp(-\Sigma_{j=1}^{J} s_j \, i_j).$$

By Theorem 4.4.2 and the above, a complete convergence result can be stated as follows.

Proposition 5.3.9 Let $\mathbb{N}_n$ be as defined in Chapter IV for the sequence $\{X_t\}$. Then $\mathbb{N}_n$ converges in distribution to some point process $\mathbb{N}$ with Laplace Transform

$$\exp\Big(-\frac{c^{\alpha}}{\Sigma_{\lambda} c^{\alpha}_{\lambda}} \iint_{\mathbb{R} \times (0,\infty)} (1 - e^{-\int_{[1,\infty)} f(s,\, tw) \psi(dw)}) ds dt\Big)$$

Where $\psi = \Sigma_{\lambda} \varepsilon_{c^{\alpha}/c^{\alpha}_{\lambda}}$.

Proof: Let ζ be a point process with the Laplace Transform described in the proposition. By Proposition 5.3.8 and Theorem 4.4.3, $\mathbb{N}_n$ converges in distribution to some point process $\mathbb{N}$. Thus it suffices to show that $\zeta \stackrel{d}{=} \mathbb{N}$. By Theorem 4.4.2 and arguments used there, we only need show

$$(5.3.22) \qquad (\zeta(\cdot \times [0, \tau_1)), \ldots, \zeta(\cdot \times [0, \tau_k))) \stackrel{d}{=} (N^{(\tau_1)}, \ldots, N^{(\tau_k)})$$

for each choice of k and $\tau_1 > \tau_2 > \ldots > \tau_k > 0$, with $(N^{(\tau_1)}, \ldots, N^{(\tau_k)})$ defined by Proposition 5.3.8. The Laplace Transform of $(\zeta(\cdot \times [0, \tau_1)), \ldots, \zeta(\cdot \times [0, \tau_k)))$ is

$$E\exp[-\Sigma_{j=1}^{k} \int_{\mathbb{R}} f_j(s) \zeta(ds\, [0, \tau_j))]$$

$$= E\exp[-\int_{\mathbb{R} \times \mathbb{R}_+} \Sigma_{j=1}^{k} f_j(s) 1_{[0,\, \tau_j)}(t) \zeta(ds \times dt)],$$

which equals the Laplace Transform of evaluated at $f(s,t)=\Sigma_{j=1}^{k} f_j(s)\cdot 1_{[0,\tau_j)}(t)$. Thus it follows from the definition of ζ that

(5.3.23) $E\exp[-\Sigma_{j=1}^{k}\int_{\mathbb{R}} f_j(s)\zeta(ds\times[0,\ \tau_j))]$

$$= \exp[-\frac{c^\alpha}{\Sigma c_\lambda}\iint_{\mathbb{R}\times\mathbb{R}_+}(1-e^{-\int[1,\infty)\Sigma_{j=1}^{k}\ f_j(s)1_{[0,\tau_j)}(tw)\psi(dw)})dsdt]$$

$$= \exp[-\frac{c^\alpha}{\Sigma\ c_\lambda}\int_{s\in\mathbb{R}}\int_{t=0}^{\tau_1}(1-e^{-\Sigma_{j=1}^{k} f_j(s)\psi[1,\ \tau_j/t)})dtds]$$

$$= \exp[-\frac{c^\alpha\tau}{\Sigma\ c_\lambda}\int_{s\in\mathbb{R}}(1-\frac{1}{\tau_1}\int_{t=0}^{\tau_1} e^{-\Sigma_{j=1}^{k}\ f_j(s)\psi[1,\ \tau_j/t)}dt)ds]$$

Simple calculations show

$$1/\tau_1\int_{t=0}^{\tau_1} e^{-\Sigma_{j=1}^{k}\ f_j(s)\psi[1,\ \tau_j/t)}dt$$

$$= 1/\tau_1\int_{t=0}^{\tau_1}\Sigma_{(i_1,\ \ldots,\ i_k)\in I_k} e^{-\Sigma_{j=1}^{k} f_j(s)i_j}\ 1_{(\psi[1,\tau_j/t)=i_j,\ j=1,\ldots,k)}(t)dt$$

$$= 1/\tau_1\sum_{(i_1,\ldots,i_k)\in I_k} e^{-\Sigma_{j=1}^{k} f_j(s)i_j}\int_0^{\tau_1} 1_{(\psi[1,\tau_j/t)=i_j,j=1,\ldots,k)}(t)dt.$$

But

$$1_{(\psi[1,\ \tau_j/t)\ =\ i_j,\ j\ =\ 1,\ \ldots,\ k)}$$

$$= 1_{(c^\alpha/c^\alpha_{\lambda_{i_j}}\ \le\ \tau_j/t\ <\ c^\alpha/c^\alpha_{\lambda_{i_j+1}},\ j\ =\ 1,\ 2,\ \ldots,\ k)}$$

$$= 1_{(\max_{1\le j\le k}(\tau_j c^\alpha_{\lambda_{i_j}}/c^\alpha)\ \le\ t\ <\ \min_{1\le j\le k}(\tau_j c^\alpha_{\lambda_{i_j+1}}/c^\alpha))},$$

showing that

$$\frac{1}{\tau_1} \int_{t=0}^{\tau_1} \exp(-\Sigma_{j=1}^{k} f_j(s)\psi[1, \tau_j/t))dt$$

$$= \sum_{(i_1, \ldots, i_k)\in I_k} \pi\{(i_1, \ldots, i_k)\}\exp(-\Sigma_{j=1}^{k} f_j(s)i_j)$$

where $\pi\{(i_1, \ldots, i_k)\} = (\min_{1\le j\le k} (\tau_j c^\alpha_{\lambda_{i_j}} /\tau_1 c^\alpha) - \max_{1\le j\le k} (\tau_j c^\alpha_{\lambda_{i_j+1}} /\tau_1 c^\alpha))^+$.

(5.3.22) follows from (5.3.23) immediately. Q. E. D.

[9] uses a more direct argument to derive a complete convergence result for the usual moving average process with regularly varying tail probabilities. It can be seen easily that the sequence of point processes $\mathbb{N}_n$ defined for the usual moving average converges in distribution to the point process $\mathbb{N}$ in the above result. This phenomenon is interesting in its own right, and is yet to be explained.

REFERENCES

[1] Adler, R. J.: Weak convergence results for extremal processes generated by dependent random variables. Ann. Probab. 6, 660-667 (1978)

[2] Berman, S. M.: Limit theorems for the maximum term in stationary sequences. Ann. Math. Statist. 35, 502-516 (1964)

[3] Berman, S. M.: A compound Poisson limit for stationary sums, and sojourns of Gaussian Processes. Ann. Probab. 8, 511-138 (1980)

[4] Berman, S. M.: Sojourns and extremes of stationary processes. Ann. Probab. 10, 1-46 (1982)

[5] Billingsley, P.: Convergence of Probability Measures. New York: Wiley (1968)

[6] Chernick, M. R.: A limit theorem for the maximum of autoregressive processes with uniform marginal distributions. Ann. Probab. 9, 145-149 (1981)

[7] Cline, D. B. H.: Infinite series of random variables with regularly varying tails. Technical Report No. 83-24, University of British Columbia (1983)

[8] Daley, D. J. and Hall, P.: Limit laws for the maximum of weighted and shifted I. I. D. random variables. Center for Stoch. Proc. Report 12, Statistics Dept. Univ. of N.C. (1983)

[9] Davis, R. and Resnick, S.: Limit theory for moving averages of random variables with regularly varying tail probabilities.

[10] Dwass, M.: Extremal processes. Ann. Math. Statist. 35, 1718-1725 (1964)

[11] Feller, W.: An Introduction to Probability Theory and its Applications, vol. 2, New York: Wiley (1971)

[12] Haan, L. de: On regular variation and its application to the weak convergence of sample extremes. Amsterdam Math. Centre Tracts 32, 1-124 (1970)

[13] Iglehart, D. L.: Weak convergence of probability measures on product spaces with applications to sums of random variables. Tech.

Rept. No. 109, Dept. of Operation Res. Stanford University (1968)

[14] Kallenberg, O.: Random Measures. Berlin: Akademie-Verlag, London-New York: Academic press (1976)

[15] Lamperti, J.: On extreme order statistics. Ann. Math. Statist. 35, 1726-1737 (1964)

[16] Leadbetter, M. R.: Point processes generated by level crossings. In: Stochastic point processes, Ed. P. A. W. Lewis. New York: Wiley (1973)

[17] Leadbetter, M. R.: On extreme values in stationary sequences. Z. Wahrsch. verw. Gebiete 28, 289-303 (1974)

[18] Leadbetter, M. R.: Weak convergence of high level exceedances by a stationary sequence. Z. Wahrsch. verw. Gebiete 34, 11-15 (1976)

[19] Leadbetter, M. R., Lindgren, G., and Rootzén, H.: Conditions for the convergence in distribution of maxima of stationary normal processes. Stochastic Process. Appl. 8, 131-139 (1978)

[20] Leadbetter, M. R.: Extremes and local dependence in stationary sequences. Z. Wahrsch. verw. Gebiete 65, 291-306 (1983)

[21] Leadbetter, M. R., Lindgren, G., Rootzén, H.: Extremes and related properties of random sequences and processes. Springer Statistics Series. Berlin-Heidelberg-New York: Springer (1983)

[22] Loynes, R. M.: Extreme values in uniformly mixing stationary stochastic processes. Ann. Math. Statist. 36, 993-999 (1965)

[23] Matthes, K., Kerstan, J., and Mecke, J.: Infinitely Divisible Point Processes. New York: Wiley (1978)

[24] Mori, T.: Limit laws for maxima and second maxima for strong-mixing processes. Ann. Probab. 4, 122-126 (1976)

[25] Mori, T.: The ralation of sums and extremes of random variables. Proc. 43rd ISI Meeting, Buenos Aires (1981)

[26] Mori, T.: Limit distributions of two-dimensional point processes generated by strong mixing sequences. Yokohama Math. J. 25, 155-168 (1977)

[27] O'Brien, G. L.: Limit theorems for the maximum term of a stationary process. Ann. Probab. 2, 540-545 (1974)

[28] O'Brien, G. L.: The limiting distribution of maxima of random variables defined on a denumerable Markov chain. Ann. Probab. 2, 103-111 (1974)

[29] Parthasarathy, K. R.: Probability Measures On Metric Spaces. New York: Academic Press (1967)

[30] Pickands, J. III: The two-dimensional Poisson process and extremal processes. J. Appl. Probab. 8, 745-756 (1971)

[31] Resnick, S. I.: Extremal processes and record value times. J. Appl. Probab. 10, 864-868 (1973)

[32] Resnick, S. I.: Weak convergence to extremal processes. Ann. Probab. 3, 951-960 (1975)

[33] Rootzén, H.: Extremes of moving averages of stable processes. Ann. Probab. 6, 847-869 (1978)

[34] Rootzén, H.: Extreme value theory for moving average processes. Center for Stoch. Proc. Report 36, Statistics Dept. Univ. of N.C. (1983)

[35] Shorrock, R. W.: On record values and record times. J. Appl. Probab. 9, 316-326 (1972)

[36] Volkonskii, V. A. and Rozanov, Yu. A.: Some limit theorems for random functions, I. Theory Probab. Appl. 4, 178-197 (1959)

[37] Welsch, R. E.: Limit laws for extreme order statistics from strong-mixing processes. Ann. Math. Statist. 43, 439-446 (1972)

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